UNIT - I

DIFFERENTIAL CALCULUS

FUNCTION:

A function f from a set D to a set E is a rule that assigns a unique element $f(x) \in E$ to each element $x \in D$.

The set D of all possible input values is called the domain of the function. The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

A symbol that represents an arbitrary no. in the domain of a function f is called an independent variable. A symbol that represents a no. in the range of f is called a dependent variable.

The function represent by four ways:

i) Verbally ii) Visually iii) Numerically

iv) Algebraically.

REAL - VALUED FUNCTIONS:

A function, whose domain and co-domain are subsets of the set of all real numbers, is known as real-valued function.

EXPLICIT FUNCTIONS:

If x and y be so related that y can be expressed explicity in terms of x, then y is called explicit funs. of x.

Eg: $y = x^2 - 4x + 2$

IMPLICIT FUNCTIONS:

If x and y be so related that y cannot be expressed explicitly in terms of x, then y is called implicit function of x.

Eg: $x^3+y^3-3xy=0$.

DOMAIN, CO-DOMAIN, RANGE AND IMAGE:

Let $f:A\to B$, then set A is called the domain of the funı. set B is called Co-domain.

The set of all the images of all the elts. of A under the funı. f is called the range of f and is denoted by $f(A)$.

Thus range of f is $f(A)=\{f(x): x\in A\}$

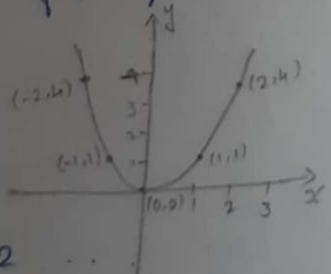
clearly, $f(A)\subseteq B$.

If $x\in A$, $y\in B$ and $y=f(x)$, then y is called the image of x under f.

GRAPH OF FUNCTIONS:

If f is a function with domain D, then its graph is the set of ordered pairs $\{(x,f(x))/x\in D\}$.

Find the domain and range and sketch the graph of the funı. $f(x)=x^2$.



Solnı. Gnı. $f(x)=x^2$ ie., $y=x^2$

Gnı. eqnı. is a parabola

Domain (x) :	0	1	-1	2	-2	..
Range (y) :	0	1	1	4	4	...

$\therefore$ The domain of f is the set of all real nosı. R. ie, $(-\infty,\infty)$

$\therefore$ The graph shows that the range is $[0,\infty)$, $[\because x^2\geq 0]$

Find the domain and the range of each function

a) $f(x)=1+x^2$ b) $f(x)=\sqrt{5x+10}$ c) $f(x)=\frac{4}{3-x}$.

Solnı.

a) Gnı. $f(x)=1+x^2 \Rightarrow y=1+x^2$

$\Rightarrow y-1=x^2$

$\Rightarrow y-1\geq 0 \Rightarrow y\geq 1$ $[\because x^2\geq 0]$

$\therefore$ The domain is $(-\infty,\infty)$ and the range is $[1,\infty)$.

b) Gnı. $f(x)=\sqrt{5x+10} \Rightarrow y=\sqrt{5x+10}$ $[\because y^2\geq 0]$

$\Rightarrow y^2=5x+10 \Rightarrow 5x+10\geq 0$

$\Rightarrow x\geq -2$

$\therefore$ The domain is $[-2,\infty)$ and the range is $[0,\infty)$.

c) Gnı. $f(x)=\frac{4}{3-x} \Rightarrow y=\frac{4}{3-x}$ division by zero is not allowed.

For $x=3$, we get $3-x=0$.

So, the domain is $(-\infty,3)\cup(3,\infty)$ and the range is $(-\infty,0)\cup(0,\infty)$.

Find the domain of each function a) $f(x)=\frac{x+4}{x^2-9}$

b) $f(x)=\sqrt[3]{2x-1}$ c) $f(x)=\frac{1}{\sqrt[4]{x^2-5x}}$.

Solnı.

a) Gnı. $f(x)=\frac{x+4}{x^2-9} \Rightarrow y=\frac{x+4}{x^2-9}$

$x^2-9=0 \Rightarrow x=\pm 3$, division by zero is not allowed.

So the domain is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

b) Gnı. $f(x)=\sqrt[3]{2x-1}$

$\Rightarrow y=\sqrt[3]{2x-1} \Rightarrow y^3=2x-1$

$\therefore$ The domain is $(-\infty,\infty)$.

c) Ans. $f(x) = \frac{1}{\sqrt[4]{x^2-5x}}$ $\Rightarrow y = \frac{1}{\sqrt[4]{x^2-5x}}$ division by zero is not allowed.

$\Rightarrow$ for $x = 0$, we get $x^2 - 5x = 0$

for $x = 5$, we get $x^2 - 5x = 0$

So, the domain is $(-\infty, 0) \cup (5, \infty)$

[$\because$ fourth root of a negative no. is not defined (as a real no.)] So, $(0,5)$ is not allowed.

PIECEWISE - DEFINED FUNCTIONS:

The functions are described by using different formula's on different parts of its domain, such funs. are called piecewise defined funs.

Find the domain and sketch the graph of the function

$$f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 1-x & \text{if } x \geq 0. \end{cases}$$

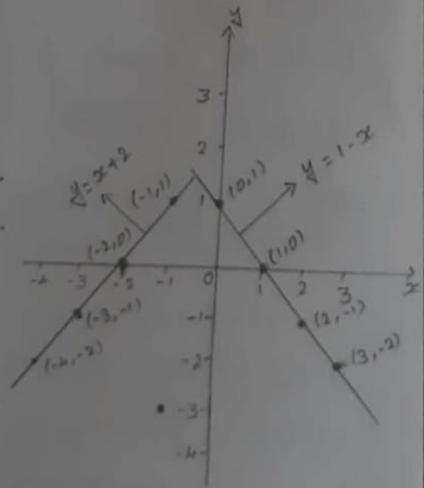
Soln.

Ans. $$f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 1-x & \text{if } x \geq 0. \end{cases}$$

ie, $y = x+2$, $x<0$

$y = 1-x$, $x \geq 0$

$x<0$	-1	-2	-3	-4	$\cdots$
$y=x+2$	1	0	-1	-2	$\cdots$
$x \geq 0$	0	1	2	3	$\cdots$
$y=1-x$	1	0	-1	-2	$\cdots$



So, the domain is $(-\infty, \infty)$.

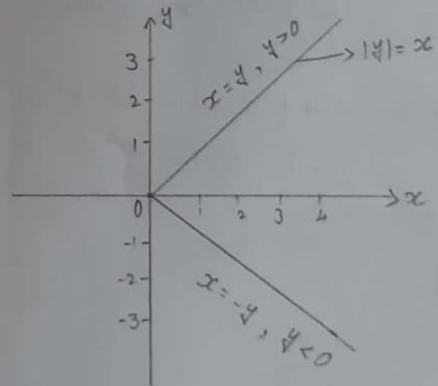
THE VERTICAL LINE TEST FOR A FUNCTION:

A curve in the xy-plane is the graph of a function of x iff no vertical line intersects the curve more than once.

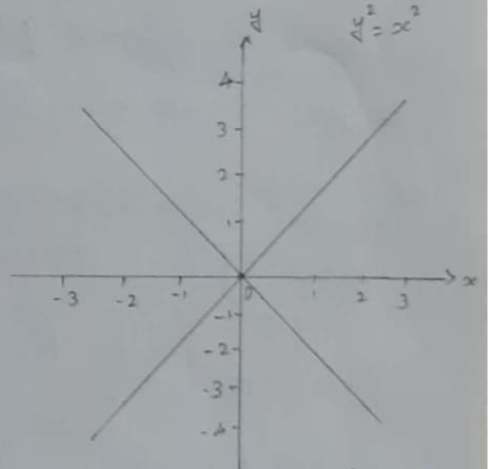
NOTE! A circle cannot be the graph of a function since some vertical lines intersect the circle twice.

Graph the following eqns. and explain they are not graphs of the funs. of x.

a) $|y| = x$ b) $y^2 = x^2$.



For each positive value of x, there are two values of y



For each value of $x \neq 0$ there are two values of y.

EVEN FUNCTION & ODD FUNCTION:

If a function $y = f(x)$ is an even function of x if $f(-x) = f(x)$, odd function of x if $f(-x) = -f(x)$ for every number x in its domain.

1) Determine whether each of the following functions is even, odd or neither even nor odd.

1) $f(x) = x^3 + x \Rightarrow$ Odd fun.

2) $f(x) = 1+3x^2-x^4 \Rightarrow f(x)$ is an even fun.

3) $f(x) = \frac{x}{x+1} \Rightarrow f(x)$ is neither even nor odd.

4) $f(x) = 1-\cos x \Rightarrow$ even fun.

5) $f(x) = x\cos x \Rightarrow$ Odd fun.

6) $f(x) = e^{x^2} \Rightarrow$ even fun.

2) Evaluate the difference quotient for the gn function

a) $f(x) = 4+3x-x^2$, $\frac{f(3+h)-f(3)}{h}$; (b) $f(x) = 1/x$, $\frac{f(x)-f(a)}{x-a}$.

Soln.

a) Gn. $f(x) = 4+3x-x^2$

$f(3+h) = 4+3(3+h)-(3+h)^2$
$= -h^2-3h+4$

$f(3) = 4$

To find: $\frac{f(3+h)-f(3)}{h}$

$\Rightarrow \frac{f(3+h)-f(3)}{h} = -(3+h)$

b) Gn. $f(x) = 1/x$ To find: $\frac{f(x)-f(a)}{x-a}$

$\Rightarrow \frac{f(x)-f(a)}{x-a} = -\frac{1}{xa}$

INCREASING AND DECREASING FUNCTIONS:

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be increasing on I.

If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is said to be decreasing on I.

Eg: * $f(x) = x^2$ is decreasing in $(-\infty, 0]$ and increasing in $[0,\infty)$

* $f(x) = -x^3$ is decreasing in $(-\infty, \infty)$.

LIMIT OF A FUNCTION:

A fun. $f(x)$ tends to a definite limit l as x tends to 'a' if the difference between $f(x)$ and l can be made as small as we like by making x approach sufficiently near 'a' and we write

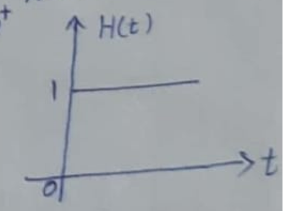
$$\lim_{x\to a} f(x) = l.$$

HEAVISIDE FUNCTION

The Heaviside fun. H is defined by $H(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t\geq 0 \end{cases}$

As t approaches 0 from the left, $\lim_{t\to 0^-} H(t) = 0$

As t approaches 0 from the right, $\lim_{t\to 0^+} H(t) = 1$



$\therefore \lim_{t\to 0} H(t)$ does not exist.

LEFT-HAND LIMIT OF $f(x)$:

The left-hand limit of $f(x)$ as x approaches a is equal to L. ie, $\lim_{x\to a^-} f(x) = L$.

Here, $x\to a^-$ means $x<a$

RIGHT-HAND LIMIT OF $f(x)$:

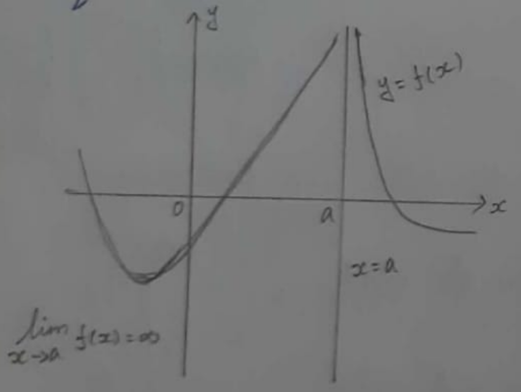
The right-hand limit of $f(x)$ as x approaches a is equal to L. $\lim_{x\to a^+} f(x) = L$.

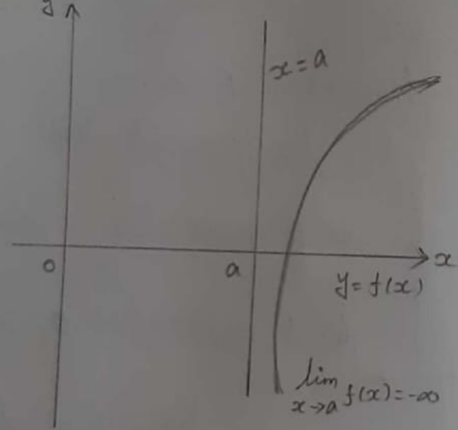
Here, $x\to a^+$ means $x>a$.

INFINITE LIMITS:

Let f be a fun. defined on both sides of a, except possibly at a itself.

* Then $\lim_{x\to a} f(x) = \infty$ means that $f(x)$ can be arbitrarily large by taking x sufficiently close to a, but not equal to a.
* Then $\lim_{x\to a} f(x) = -\infty$ means that $f(x)$ can be arbitrarily large negative by taking x sufficiently close to a, but not equal to a.





VERTICAL ASYMPTOTE:

The line $x = a$ is called a vertical asympote of the curve $y = f(x)$ if atleast one of the following statement is true:

$\lim_{x\to a} f(x) = \infty$ $\quad \lim_{x\to a^-} f(x) = \infty$ $\quad \lim_{x\to a^+} f(x) = \infty$

$\lim_{x\to a} f(x) = -\infty$ $\quad \lim_{x\to a^+} f(x) = -\infty$ $\quad \lim_{x\to a^-} f(x) = -\infty$.

1) Guess the value of $\lim_{x\to 0} \frac{\sin x}{x}$.

Soln. Gn. $\lim_{x\to 0} \frac{\sin x}{x}$, Here, $f(x) = \frac{\sin x}{x}$.

x	$f(x)$	x	$f(x)$
± 1.0	0.84147098	± 0.1	0.99833417
± 0.5	0.95885108	± 0.05	0.99958339
± 0.4	0.97354586	± 0.01	0.99998333
± 0.3	0.98506736	± 0.005	0.99999583
± 0.2	0.99334665	± 0.001	0.99999983

From the table, $\lim_{x\to 0} f(x) = 1$.

2) Determine the infinite limit

i) $\lim_{x\to 1} \frac{2-x}{(x-1)^2}$ (ii) $\lim_{x\to -3^+} \left(\frac{x+2}{x+3}\right)$ (iii) $\lim_{x\to 5^-} \frac{e^x}{(x-5)^3}$

Soln.

i) $\lim_{x\to 1} \frac{2-x}{(x-1)^2} = \frac{1}{0} = \infty$

ii) Gn. $\lim_{x\to -3^+} \left(\frac{x+2}{x+3}\right)$

$x \to -3^+ \Rightarrow x$ is close to -3 but larger than -3.

Nr $= x+2$ becomes negative

Dr $= x+3$ becomes positive

Let $x = -2.9$

Nr $= -2.9 + 2 = -0.9 = -$ve

Dr $= -2.9 + 3 = 0.1 = +$ve

$\therefore \lim_{x\to -3} \frac{x+2}{x+3} = -\infty$

iii) Gn. $\lim_{x\to 5^-} \frac{e^x}{(x-5)^3}$

$x \to 5^- \Rightarrow x$ is close to 5 but smaller than 5.

Nr $= e^x$ becomes positive

Dr $= (x-5)^3$ becomes negative

Let $x = 4.9$

$e^{4.9} = +$ve

$(4.9-5)^3 = -$ve.

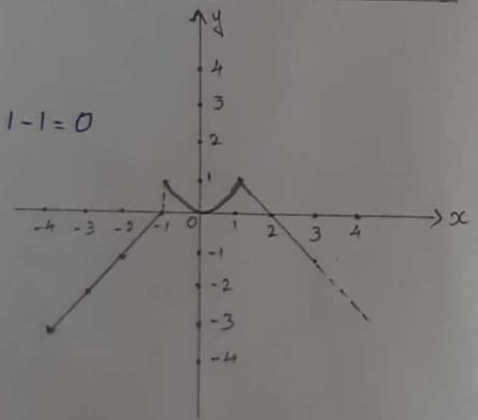
$\therefore \lim_{x\to 5^-} \frac{e^x}{(x-5)^3} = -\infty$

3) Sketch the graph of the function $f(x) = \begin{cases} 1+x & ; \; x<-1 \\ x^2 & ; \; -1 \le x \le 1 \\ 2-x & ; \; x \ge 1 \end{cases}$

and use it to determine the values of 'a' for which $\lim_{x\to a} f(x)$ exists.

Soln:

	$1+x$; $x<-1$			x^2 ; $-1\le x\le 1$			$2-x$; $x\ge 1$			
x	-2	-3	-4	-1	0	1	1	2	3	4
$f(x)$	-1	-2	-3	1	0	1	1	0	-1	-2



At $x=-1$

$f(-1^-) = \lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (1+x) = 1-1 = 0$

$f(-1) = \lim_{x\to -1} f(x) = \lim_{x\to -1} x^2 = 1$

$f(-1^+) = \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x^2 = 1$

Here, $f(-1^-) \ne f(-1) = f(-1^+)$

$\therefore f$ is discontinuous at $x=-1$.

At $x=1$:

$f(1^-) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$

$f(1) = \lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 = 1$

$f(1^+) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 1$

Here, $f(1^-) = f(1) = f(1^+)$

$\therefore f$ is Continuous at $x=1$.

Hence, $\lim_{x\to a} f(x)$ exists for all 'a' except at '$a=-1$'.

H.W

1) Sketch the graph of the function

$$f(x) = \begin{cases} 1+\sin x & \text{if } x<0 \\ \cos x & \text{if } 0\le x\le \pi \\ \sin x & \text{if } x>\pi \end{cases}$$

and use it to determine the value of 'a' for which $\lim_{x\to a} f(x)$ exists.

4) Discuss the behaviour of the following as $x\to 0$

i) $U(x) = \begin{cases} 0 & , \; x<0 \\ 1 & , \; x\ge 0 \end{cases}$ (ii) $f(x) = \begin{cases} 1/x & , \; x\ne 0 \\ 0 & , \; x=0 \end{cases}$ (iii) $h(x) = \begin{cases} 0 & , \; x\le 0 \\ \sin 1/x & , \; x>0 \end{cases}$

Soln-

i) The unit step fun. $U(x)$ has no limit as $x\to 0$ because its values jump at $x=0$.

For -ve values of x arbitrary close to zero, $U(x)=0$.

For +ve values of x arbitrary close to zero, $U(x)=1$.

$\therefore$ There is no single value approached by $U(x)$ as $x\to 0$.

ii) $f(x)$ has no limit as $x\to 0$ because the values of f grow arbitrarily large in absolute value as $x\to 0$.

~~iii)~~ The values do not stay close to any fixed real no.

iii) $h(x)$ has no limit as $x\to 0$ because the funs. values oscillate b/w +1 and -1 in every open interval containing 0.

The values do not stay close to any one no. as $x\to 0$.

H.W 2) Guess the value of the limit (if it exists) by evaluating the fun. at the gn. nos. (correct to six decimal places).

i) $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$, $x=\pm 1, \pm 0.5, \pm 0.1, \pm 0.05, \pm 0.01$ $\Rightarrow \frac{1}{2}$

ii) $\lim_{x\to 1} \frac{x-1}{x^2-1}$ $\Rightarrow 0.5$

3) Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

$$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} \Rightarrow \frac{1}{4}.$$

CALCULATING LIMITS USING THE LIMIT LAWS:

LIMIT LAWS:

Suppose that c is a constant and the limits

$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

Then

* $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
* $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
* $\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x)$
* $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
* $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$.
* $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a +ve integer.
* $\lim_{x \to a} c = c$ * $\lim_{x \to a} x = a$ * $\lim_{x \to a} x^n = a^n$, where n is +ve integer.
* $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is +ve integer

(If n is even, we assume that $a > 0$)

DIRECT SUBSTITUTION PROPERTY: If f is a polynomial or a rational fun. and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

1) Evaluate the following limits and justify each step.

a) $\lim_{x \to -2} [3x^4 + 2x^2 - x + 1] = 3 \lim_{x \to -2} x^4 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} x + \lim_{x \to -2} 1$

$= 59$

b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$

c) $\lim_{u \to -2} \sqrt{u^4 + 3u + 6} = 4.$

LIMITS OF POLYNOMIALS:

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_0.$$

LIMITS OF RATIONAL FUNCTIONS:

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

1) Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

Soln.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = 2.$$

2) Find $\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2}$.

Soln.

$$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3}$$

$$= \lim_{t \to 0} \frac{(t^2+9) - 9}{t^2(\sqrt{t^2+9} + 3)} = \frac{1}{6}$$

3) Find $\lim_{t \to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t}$

$$= \lim_{t \to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t} + \sqrt{1-t}}{\sqrt{1+t} + \sqrt{1-t}} = 1$$

4) Find $\lim_{t\to 0}\left[\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right]$

$= \lim_{t\to 0}\frac{1}{t}\left[\frac{1-\sqrt{1+t}}{\sqrt{1+t}}\right] = -\frac{1}{2}$.

5) P.T $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Soln. Let $f(x) = \frac{|x|}{x}$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x} = \lim_{x\to 0^+} 1 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{(-x)}{x} = -1$

Here, $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

$\therefore \lim_{x\to 0}\frac{|x|}{x}$ does not exist.

6) The signum (or sign) fun., denoted by sgn, is defined by

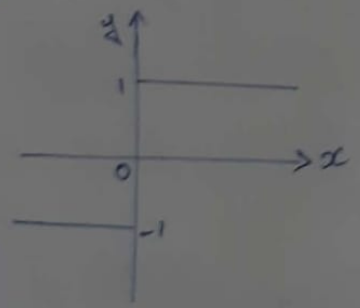
$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$

i) Sketch the graph of this fun. (ii) Find each of the following limits or explain why it does not exist.

a) $\lim_{x\to 0^+}\operatorname{sgn} x$ (b) $\lim_{x\to 0^-}\operatorname{sgn} x$ (c) $\lim_{x\to 0}\operatorname{sgn} x$ (d) $\lim_{x\to 0}|\operatorname{sgn} x|$

Soln.

i)



ii) a) $\lim_{x\to 0^+}\operatorname{sgn} x = 1$ $[\because x>0]$

b) $\lim_{x\to 0^-}\operatorname{sgn} x = -1$ $[\because x<0]$

c) $\lim_{x\to 0^+}\operatorname{sgn} x \neq \lim_{x\to 0^-}\operatorname{sgn} x$

So, $\lim_{x\to 0}\operatorname{sgn} x$ does not exist.

d) $\lim_{x\to 0}|\operatorname{sgn} x| = 1$ $\left[\because \lim_{x\to 0^+}|\operatorname{sgn} x| = \lim_{x\to 0^+}[\operatorname{sgn} x] = 1\right.$

$\lim_{x\to 0^-}|\operatorname{sgn} x| = \lim_{x\to 0^-}(-\operatorname{sgn} x) = 1$

$\Rightarrow \lim_{x\to 0^+}|\operatorname{sgn} x| = \lim_{x\to 0^-}|\operatorname{sgn} x|$

H.W

7) Let $g(x) = \frac{x^2+x-6}{|x-2|}$. (a) Find (i) $\lim_{x\to 2^+} g(x)$ (ii) $\lim_{x\to 2^-} g(x)$

(b) Does $\lim_{x\to 2} g(x)$ exist?

Soln.

a) $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}\frac{(x-2)(x+3)}{x-2} = 5$

$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}\frac{(x-2)(x+3)}{-(x-2)} = -5$

b) $\lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$.

$\therefore \lim_{x\to 2} g(x)$ does not exist.

8) If $\lim_{x\to 0}\frac{f(x)}{x^2} = 5$, find the following limits.

a) $\lim_{x\to 0} f(x)$ b) $\lim_{x\to 0}\frac{f(x)}{x}$.

Soln.

a) Gn. $\lim_{x\to 0}\frac{f(x)}{x^2} = 5$

$\Rightarrow \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} x^2} = 5 \Rightarrow \lim_{x\to 0} f(x) = 5\lim_{x\to 0} x^2 = 0$

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b) Ans- $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$

$\Rightarrow \lim_{x\to 0} \frac{f(x)}{x} \cdot \lim_{x\to 0} \frac{1}{x} = 5$

$\Rightarrow \lim_{x\to 0} \frac{f(x)}{x} = 5 \lim_{x\to 0} x = 5(0) = 0.$

NOTE:

$[[x]]$ means the largest integer that is less than or equal to x

Eg: $[[4]] = 4$, $[[5.8]] = 5$, $[[\pi]] = 3$, $[[\sqrt{2}]] = 1$, $[[-\frac{1}{2}]] = -1$.

9) Find if $[[x]]$ denotes the greatest integer that is less than or equal to x.

a) $\lim_{x\to -2^+} [[x]]$ b) $\lim_{x\to -2} [[x]]$ c) $\lim_{x\to -2.4} [[x]]$

d) $\lim_{x\to n^-} [[x]]$, n is an integer e) $\lim_{x\to n^+} [[x]]$, n is an integer

f) For what values of a, does $\lim_{x\to a} [[x]]$ exist?

Soln.

w.k.T, $[[x]] = -2$ for $-3 \le x \le -2$.

a) $\lim_{x\to -2^+} [[x]] = \lim_{x\to -2^+} (-2) = -2 \longrightarrow (1)$

b) $\lim_{x\to 2^-} [[x]] = \lim_{x\to 2^-} (-3) = -3 \longrightarrow (2)$

From (1)&(2), $(1) \ne (2)$

$\therefore \lim_{x\to -2} [[x]]$ does not exist.

c) $\lim_{x\to -2.4} [[x]] = \lim_{x\to -2.4} (-3) = -3$

d) $\lim_{x\to n^-} [[x]] = \lim_{x\to n^-} (n-1) = n-1$

e) $\lim_{x\to n^+} [[x]] = \lim_{x\to n^+} n = n.$

f) a is not an integer.

Theorem:

If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

The Squeeze Theorem (or) Sandwich Theorem or The Pinching Theorem:

If $f(x) \le g(x) \le h(x)$ when x is near a and

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

1) Use the squeeze thm., find the value of (a) $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$

(b) $\lim_{x\to 0} \sqrt{x^3+x^2} \sin(\frac{\pi}{x})$

Soln.

a) w.k.T Multiply by x^2

$-1 \le \sin(\frac{1}{x}) \le 1 \Rightarrow -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$

$\lim_{x\to 0} (-x^2) = \lim_{x\to 0} (x^2) = 0$

$\Rightarrow \lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$ [by squeeze thm.]

b) w.k.T

$-1 \le \sin \frac{\pi}{x} \le 1$

Multiply by $\sqrt{x^3+x^2}$

$\Rightarrow -\sqrt{x^3+x^2} \le \sqrt{x^3+x^2} \sin \frac{\pi}{x} \le \sqrt{x^3+x^2}$

$\lim_{x\to 0} [-\sqrt{x^3+x^2}] = \lim_{x\to 0} \sqrt{x^3+x^2} = 0.$

$\Rightarrow \lim_{x\to 0} \sqrt{x^3+x^2} \sin(\frac{\pi}{x}) = 0$ (by squeeze thm.)

CONTINUITY:

A fun- f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

NOTE:

* If f is continuous at a, then
  - i) $f(a)$ should exist (that is, a is in the domain of f)
  - ii) $\lim_{x \to a} f(x)$ exists both on the left and right
  - iii) $\lim_{x \to a} f(x) = f(a)$.

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a.

* The fun- $f(x)$ is said to be discontinuous at $x = a$ if one or more of the above three conditions are not satisfied.

Defn-

A function f is continuous from the right at a no- 'a' if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

NOTE:

A function is continuous on an interval if it is continuous at every no- in the interval.

Theorem 1 If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

i) $f+g$ (ii) $f-g$ (iii) cf (iv) fg (v) f/g if $g(a) \neq 0$.

Theorem: 2

a) Any polynomial is continuous everywhere; that is, it is continuous on $R = (-\infty, \infty)$.

b) Any rational fun- is continuous whenever it is defined; that is, it is continuous on its domain.

Theorem: 3

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b).$$

In other words, $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$

Theorem: 4

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Theorem: 5 The Intermediate Value Theorem:

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any no- between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$ then there exists a no- c in (a, b) such that $f(c) = N$.

1) Explain why the fun. is discontinuous at the gn. no. 'a'

a) $f(x) = \frac{1}{x+2}$, $a=-2$ (b) $f(x) = \begin{cases} e^x, & x<0 \\ x^2, & x \geq 0 \end{cases}$, $a=0$

c) $f(x) = \begin{cases} \cos x, & x<0 \\ 0, & x=0 \\ 1-x^2, & x>0 \end{cases}$, $a=0$.

Soln.

a) Gn. $f(x) = \frac{1}{x+2}$, $a=-2$

$f(-2) = \frac{1}{0} = \infty$, undefined

$\therefore$ $f(x)$ is discontinuous at the gn no. 'a'

b) Gn. $f(x) = \begin{cases} e^x, & x<0 \\ x^2, & x \geq 0 \end{cases}$, $a=0$.

$\lim_{x \to 0} f(x) = \lim_{x \to 0} e^x$ does not exist.

$\therefore$ $f(x)$ is discontinuous at the gn. no. 'a'.

c) Gn. $f(x) = \begin{cases} \cos x, & x<0 \\ 0, & x=0 \\ 1-x^2, & x>0 \end{cases}$, $a=0$

Here, $f(0)=0$ is defined and $\lim_{x \to 0} f(x) = \lim_{x \to 0} \cos x = 1$

$\lim_{x \to 0} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at the gn no. 'a'.

HW

2) Explain why the fun. is discontinuous at the gn. no. 'a'.

a) $f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3}, & x \neq 3 \\ 6, & x=3 \end{cases}$, $a=3$.

b) $f(x) = \begin{cases} 1/x+2, & x \neq -2 \\ 1, & x=-2 \end{cases}$

3) How would you "remove the discontinuity" of f? In other words, how would you define $f(2)$ in order to make f continuous at 2?

a) $f(x) = \frac{x^3-8}{x^2-4}$ b) $f(x) = \frac{x^2-7x+10}{x-2}$.

Soln.

a) Gn. $f(x) = \frac{x^3-2^3}{x^2-2^2} = \frac{(x-2)(x^2+2x+2^2)}{(x-2)(x+2)}$

$f(2) = 3$

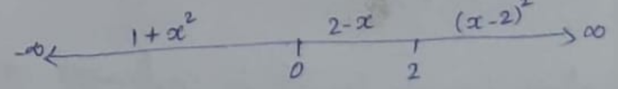
b) Gn. $f(x) = \frac{x^2-7x+10}{x-2}$

$= \frac{(x-5)(x-2)}{x-2} \Rightarrow f(2) = -3$

4) Find the domain where the function f is continuous. Also find the nos. at which the fun. f is discontinuous, where

$f(x) = \begin{cases} 1+x^2, & x \leq 0 \\ 2-x, & 0<x \leq 2 \\ (x-2)^2, & x>2 \end{cases}$

Soln.



At $x=0$

$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} (1+x^2) = 1$ $\longrightarrow (1)$

$f(0^-) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1+x^2) = 1$ $\longrightarrow (2)$

$f(0^+) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2-x) = 2$ $\longrightarrow (3)$

From (1), (2) & (3)

$f(0^-) = f(0) \neq f(0^+)$

So, f is continuous on the left at $x=0$.

f is discontinuous on the right at $x=0$.

Hence, f is discontinuous at $x=0$.

At $x=2$:

$$f(2)=\lim_{x\to 2} f(x)=\lim_{x\to 2}(2-x)=0 \longrightarrow (3)$$

$$f(2^-)=\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2-x)=0 \longrightarrow (4)$$

$$f(2^+)=\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(x-2)^2=0 \longrightarrow (5)$$

From (3), (4) & (5)

$$f(2^-)=f(2)=f(2^+)$$

Hence, f is continuous at $x=2$.

The domain of f is $(-\infty,0)\cup(0,\infty)$

H.W

5) Find the nos. that at which f is discontinuous, At which of these nos. if f continuous from the right from the left or neither? When. $f(x)=\begin{cases} x+2, & x<0 \\ e^x, & 0\le x\le 1 \\ 2-x, & x>1 \end{cases}$

1) S.T the function $f(x)=1-\sqrt{1-x^2}$ is continuous on the interval $[-1,1]$.

Soln.

$$\lim_{x\to -1^+} f(x)=1=f(-1)$$

$$\lim_{x\to 1^-} f(x)=1=f(1)$$

So, f is continuous from the right at -1 and f is continuous from the left at 1.

$\therefore$ f is continuous on $[-1,1]$

2) Where are the following fun. continuous? $f(x)=\sin(x^3)$

Soln.

we have $h(x)=f(g(x))$, where

$g(x)=x^3$ and $f(x)=\sin x$.

Now, g is continuous on R, Since it is a polynomial, and f is also continuous everywhere. Thus $h=f\circ g$ is continuous on R.

3) Suppose f and g are continuous functions such that $g(2)=6$ and $\lim_{x\to 2}[3f(x)+f(x)g(x)]=36$. Find $f(2)$.

Soln.

Gn. $\lim_{x\to 2}[3f(x)+f(x)g(x)]=36$, $g(2)=6$.

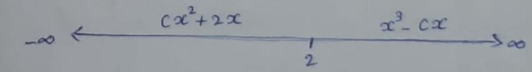
$$\Rightarrow 3\lim_{x\to 2} f(x)+\lim_{x\to 2} f(x)g(x)=36$$

$$\Rightarrow 3f(2)+f(2)g(2)=36$$

$$\Rightarrow f(2)=4$$

4) For what value of the constant c is the fun. f continuous at $(-\infty,\infty)$. $f(x)=\begin{cases} cx^2+2x, & x<2 \\ x^3-cx, & x\ge 2 \end{cases}$

Soln.



At $x=2$

Gn. f is continuous $\Rightarrow f(2^-)=f(2)=f(2^+) \longrightarrow (1)$

$$f(2)=\lim_{x\to 2} f(x)=\lim_{x\to 2}(x^3-cx)=8-2c$$

$$f(2^-)=\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}[cx^2+2x]=4c+4$$

$(1)\Rightarrow$ $4c+4=8-2c$

$$\Rightarrow c=\tfrac{2}{3}$$

H.W

5) Where the function $f(x) = \dfrac{\log x + \tan^{-1} x}{x^2 - 1}$ continuous?

6) If $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x<2 \\ ax^2 - bx + 3, & 2 \le x < 3 \\ 2x - a + b, & x \ge 3 \end{cases}$ is continuous for all real x, find the values of a and b.

7) Evaluate $\lim_{x\to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right)$

Soln:-

W.K.T Arc sin is a continuous function.

$$\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right)$$

$$= \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right) = \pi/6.$$

DERIVATIVES:

TANGENT LINE:

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ provided that this limit exists.

(or) $m = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$.

DERIVATIVE:

The derivative of a function f at a number a, denoted by $f'(a)$, is $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ if this limit exists.

(or)

$f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x-a}$, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

NOTE: The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

i.e., $y - f(a) = f'(a)(x-a)$.

1) Find,

a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.

b) Find eqns. of the tangent lines at the points $(1,1)$ and $(4, \frac{1}{2})$.

Soln:-

a) Gn. $y = 1/\sqrt{x} = x^{-1/2} \Rightarrow y' = \dfrac{-1}{2x^{3/2}}$

$\therefore$ Slope $(m) = [y']_{x=a} = -\dfrac{1}{2}\left(\dfrac{1}{a\sqrt{a}}\right)$

b) (i) At $(1,1)$

$m = (y')_{(1,1)} = -1/2$

W.K.T

The eqn. of the tangent line at the point (x_1, y_1) is

$$(y - y_1) = m(x - x_1)$$

$\Rightarrow y - 1 = -1/2\,(x-1) \Rightarrow y = -1/2\, x + 3/2.$

(ii) $m = (y')_{(4, 1/2)} = -1/16$

$\therefore$ The eqn. of the tangent line at the pt. $(4, \frac{1}{2})$ is

$$y - 1/2 = -1/16\,(x-4)$$

$$\Rightarrow y = -\frac{1}{16}x + \frac{3}{4}.$$

2) If $f(x) = x^3 - x$, then find $f'(x)$ & $f''(x)$.

Soln:- Gn. $f(x) = x^3 - x$

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= 3x^2 - 1$

$$f''(x) = (f')'(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h\to 0} \frac{[3(x+h)^2-1] - [3x^2-1]}{h} = 6x.$$

3) If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

Soln: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right]$$

$$= \lim_{h\to 0} \frac{(x+h)-x}{h[\sqrt{x+h}+\sqrt{x}]} = \frac{1}{2\sqrt{x}}$$

Here, $f'(x)$ exists if $x>0$, So the domain of f' is $(0,\infty)$.

This is smaller than the domain of f, which is $[0,\infty)$

DIFFERENTIATION RULES:

Derivatives of Polynomials:

Formulae:

* Derivative of a constant fun. is zero. ie, $\frac{d}{dx}(c) = 0$.

* The Power rule:

If n is any real no. then $\frac{d}{dx}(x^n) = nx^{n-1}$

* The constant multiple rule:

$$\frac{d}{dx}[c\,f(x)] = c\frac{d}{dx}[f(x)]$$

* The Sum rule:

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

* The difference rule:

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

* Equation of tangent line is $(y-y_1) = m(x-x_1)$ where $m = \frac{dy}{dx}$

* Equation of normal line is $(y-y_1) = -\frac{1}{m}(x-x_1)$; $m = \frac{dy}{dx}$

1) Differentiate the following function.

i) $f(x) = x^{1000}$

Ans- $f(x) = x^{1000} \Rightarrow f'(x) = 1000\,(x)^{999}$

ii) $y = \sqrt[3]{x^2}$

Ans- $y = x^{2/3} \Rightarrow y' = \frac{2}{3}x^{-1/3}$

iii) $y = \sqrt{x^{2+\pi}}$

Ans- $y = \sqrt{x^{2+\pi}} = (x^{2+\pi})^{1/2}$

$y' = \frac{1}{2}(x^{2+\pi})^{-1/2} \cdot (2+\pi)\,x^{2+\pi-1} = (1+\frac{\pi}{2})\,x^{\pi/2}$ //

iv) $y = \frac{x^2+4x+3}{\sqrt{x}}$

$y' = \frac{3}{2}\sqrt{x} + 2\cdot\frac{1}{\sqrt{x}} - \frac{3}{2}x^{-3/2}$

v) $y = x^{2.4} + e^{2.4}$

$y' = 2.4\,x^{1.4}$ $\qquad [\because e^{2.4} = \text{constant}]$

2) Find the first and second derivatives of the following funs.

i) $f(x) = 10x^{10} + 5x^5 - x$ (ii) $f(x) = 2x - 5x^{3/4}$

Soln.

i) $f'(x) = 100x^9 + 25x^4 - 1$

$f''(x) = 900x^8 + 100x^3$

(ii) $f'(x) = 2 - \frac{15}{4} x^{-1/4}$

$f''(x) = \frac{15}{16} x^{-5/4}$

3) The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find

a) the velocity and acceleration as funs. of t.

b) the acceleration after 2s, and

c) the acceleration when the velocity is 0.

Soln. Given. $s = t^3 - 3t$

a) $v = \frac{ds}{dt} = 3t^2 - 3$

$a = \frac{dv}{dt} = 6t$

b) If the acceleration after 2 seconds, then

$$\left(\frac{dv}{dt}\right)_{t=2} = 12 \ m/s^2$$

c) the acceleration when the velocity is 0

$v = 0 \Rightarrow 3t^2 - 3 = 0$

$\Rightarrow t = \pm 1$

[Reject $t = -1$]

$$\left(\frac{dv}{dt}\right)_{t=1} = 6 \ m/s^2$$

5) Find the pts on the curve $y = x^4 - 6x^2 + 8$ where the tangent line is horizontal.

Soln. Horizontal tangents occur where the derivative is zero. ie., $\frac{dy}{dx} = 0$

$4x^3 - 12x = 0$

$\Rightarrow x = 0, \ x = \pm\sqrt{3}$

x	0	$\sqrt{3}$	$-\sqrt{3}$
y	8	-1	-1

The corresponding points are $(0, 8)$, $(\sqrt{3}, -1)$ & $(-\sqrt{3}, -1)$.

4) Find a second degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, $P''(2) = 2$.

Soln. Given. $P(2) = 5$, $P'(2) = 3$, $P''(2) = 2$.

Let $P(x) = ax^2 + bx + c \longrightarrow (1)$

$P'(x) = 2ax + b \longrightarrow (2)$

$P''(x) = 2a \longrightarrow (3)$

Given. $P''(2) = 2 \Rightarrow a = 1$

$P'(2) = 3 \Rightarrow b = -1$

$P(2) = 5 \Rightarrow c = 3$.

$\therefore (1) \Rightarrow P(x) = x^2 - x + 3$.

5) Evaluate $\lim\limits_{x \to 1} \frac{x^{1000} - 1}{x - 1}$

Soln. W.K.T

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} = \lim_{x \to 1} \frac{x^{1000} - 1^{1000}}{x - 1} = f'(1) \longrightarrow (1)$$

where $f(x) = x^{1000} \Rightarrow f'(x) = 1000x^{999} \Rightarrow f'(1) = 1000$

$$\therefore (1) \Rightarrow \lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} = 1000.$$

DERIVATIVES OF EXPONENTIAL FUNCTIONS:

Derivative of the Natural Exponential fun. $\frac{d}{dx}(e^x) = e^x$.

1) $y = 3e^x + \frac{4}{\sqrt[3]{x}}$

Given. $y = 3e^x + 4x^{-1/3} \Rightarrow y' = 3e^x - \frac{4}{3} x^{-4/3}$

2) $y = 2^x$

Given. $y = 2^x = e^{x \log 2}$

$y' = 2^x \log 2$.

3) $y = e^x - x^3$

$y' = e^x - 3x^2$

The Product And Quotient Rule:

The Product Rule:

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

ie., $\frac{d}{dx}[uv] = uv' + vu'$

The Quotient Rule:

If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

ie., $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$.

1) Differentiate the following functions:

i) $f(x) = (1-e^x)(x+e^x)$

Soln.
Gn. $f(x) = (1-e^x)(x+e^x)$

$$f'(x) = (1-e^x)\frac{d}{dx}(x+e^x) + (x+e^x)\frac{d}{dx}(1-e^x)$$

$$= 1 - 2e^{2x} - xe^x.$$

2) If $f(x) = \sqrt{x}\, g(x)$, where $g(4) = 2$ and $g'(4) = 3$ find $f'(4)$.

Soln.
Gn. $f(x) = \sqrt{x}\, g(x)$

$$f'(x) = \sqrt{x}\, g'(x) + g(x)\frac{1}{2\sqrt{x}}$$

$$f'(4) = \frac{13}{2} = 6.5 \qquad [\because g(4) = 2 \text{ \& } g'(4) = 3]$$

3) Find eqns. of the tangent line and normal line to the given curve at the specified point $f(x) = 2xe^x$, $(0,0)$.

Soln.
Gn. $f(x) = 2xe^x$

$$f'(x) = 2x\frac{d}{dx}e^x + e^x\frac{d}{dx}2x = 2xe^x + e^x(2)$$

$$m = (f'(x))_{(0,0)} = 2e^0(0+1) = 2$$

The eqn. of tangent line is $y - y_1 = m(x - x_1)$

$$y - 0 = 2(x-0) \Rightarrow y = 2x$$

The eqn. of normal line is $(y - y_1) = -\frac{1}{m}(x - x_1)$

$$y - 0 = -\frac{1}{2}(x-0) \Rightarrow 2y = -x \Rightarrow y = -\frac{1}{2}x.$$

H.W

4) The curve $y = \frac{x}{1+x^2}$ is called a serpentine. Find an eqn. of the tangent line to this curve at the point $(3, 0.3)$.

$$\Rightarrow y = -0.08x + 0.54$$

5) Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$ and $g'(5) = 2$. Find

i) $(fg)'(5)$ (ii) $\left(\frac{f}{g}\right)'(5)$.

6) If $f(x) = \frac{A}{B + Ce^x}$ then find $f'(x)$. $\Rightarrow f'(x) = \frac{-ACe^x}{(B+Ce^x)^2}$

7) Find the derivative of the following funs.

i) $f(x) = (x - \sqrt{x})(x + \sqrt{x})$

$\Rightarrow f'(x) = 2x - 1$

(ii) $f(x) = e^x(x + x\sqrt{x})$.

$\Rightarrow f'(x) = e^x(1 + \frac{3}{2}\sqrt{x} + x + x\sqrt{x}$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS:

Formulae:

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

NOTE:

1. $\frac{d}{dx}(\sin mx) = m \cos mx$
2. $\tan x = \frac{\sin x}{\cos x}$
3. $\cot x = \frac{\cos x}{\sin x}$
4. $\sec x = \frac{1}{\cos x}$
5. $\operatorname{cosec} x = \frac{1}{\sin x}$
6. $1 + \tan^2 x = \sec^2 x$
7. $\sin^2 x + \cos^2 x = 1$
8. $1 + \cot^2 x = \operatorname{cosec}^2 x$.

1) Find the derivatives of the following funs!.

i) $y = \frac{\sec x}{1 + \tan x}$

Soln.

$$y' = \frac{\sec x [\tan x - 1]}{(1 + \tan x)^2}$$

ii) $y = \operatorname{cosec} x + e^x \cot x$

$\Rightarrow y' = -\operatorname{cosec} x \cot x - e^x \operatorname{cosec}^2 x + e^x \cot x$.

iii) $f(x) = x e^x \operatorname{cosec} x$.

w.K.T

$$\frac{d}{dx}[f(x) g(x) h(x)] = f(x) g(x) h'(x) + f(x) g'(x) h(x) + f'(x) g(x) h(x)$$

$$\Rightarrow f'(x) = e^x \operatorname{cosec} x [x + 1 - x \cot x]$$

2) An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. Its position at time t is $s = f(t) = 4 \cos t$.
Find the velocity and accelaration at time t and use them to analyze the motion of the object.

Soln.

The velocity

$$V = \frac{ds}{dt} = -4 \sin t$$

The acceleration

$$a = \frac{dv}{dt} = -4 \cos t.$$

The object oscillates from the lowest point ($s = 4$ cm) to the highest point ($s = -4$ cm).

The period of the oscillation is 2π, the period of $\cos t$.

The speed is $|v| = 4|\sin t|$, which is greatest when $|\sin t| = 1$ that is, when $\cos t = 0$.

So, the object moves fastest as it passes through its equilibrium position ($s = 0$).

Its speed is 0 when $\sin t = 0$, that is, at the high and low points.

The acceleration $a = -4 \cos t = 0$ when $s = 0$.

It has greatest magnitude at the high and low points.

3) Find the 25^{th} derivative of $\cos x$.

Soln. Gn:- $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{IV}(x) = \cos x$

$\vdots$

$f^{(n)}(x) = \cos x$, whenever n is a multiple of 4

$\therefore f^{(24)}(x) = \cos x$

$\Rightarrow f^{(25)}(x) = -\sin x$

H.W

4) Find the derivatives of $y = \frac{\cos x}{1 - \sin x}$ & $y = \frac{x \sin x}{1+x}$.

5) Find an eqn of the tangent line to the curve $y = 2x \sin x$ at the point $(\frac{\pi}{2}, \pi)$.

6) Find $\lim_{x \to 0} x \cot x$

7) Find $\frac{d^{99}}{dx^{99}} (\sin x)$

THE CHAIN RULE:

1) Find the derivatives of the following:

i) $y = (1-x^2)^{10}$ (ii) $y = \sqrt[4]{1+2x+x^3}$ (iii) $y = \sec(\tan x)$

iv) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

Soln:-

i) Gn:- $y = (1-x^2)^{10}$ $\longrightarrow (1)$

Put $u = 1-x^2$, $\frac{du}{dx} = -2x$

$(1) \Rightarrow y = u^{10} \Rightarrow \frac{dy}{du} = 10u^9$

By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = (10u^9)(-2x) = -20x(1-x^2)^9$$

ii) Gn:- $y = \sqrt[4]{1+2x+x^3}$

$\Rightarrow y = (1+2x+x^3)^{1/4}$

$$\frac{dy}{dx} = \frac{1}{4}(1+2x+x^3)^{-3/4}(2+3x^2)$$

iii) Gn:- $y = \sec(\tan x)$

$$\frac{dy}{dx} = \sec(\tan x)\tan(\tan x)\frac{d}{dx}(\tan x)$$

$$= \sec(\tan x)\tan(\tan x)\sec^2 x.$$

iv) Gn:- $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

$$\frac{dy}{dx} = \frac{1}{2(\sqrt{x+\sqrt{x+\sqrt{x}}})}\left[1 + \frac{1}{2(\sqrt{x+\sqrt{x}})}\left[1+\frac{1}{2\sqrt{x}}\right]\right]$$

2) If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$. find $F'(0)$.

Soln: Gn:- $F(x) = f(3f(4f(x)))$

$$F'(x) = f'(3f(4f(x)))\, 3f'(4f(x))\, 4f'(x)$$

$$F'(0) = 96.$$

v) $y = 2^{\sin \pi x}$

Soln:- Gn:- $y = 2^{\sin \pi x}$

$= e^{\sin \pi x \log 2}$

$$\frac{dy}{dx} = e^{\sin \pi x \log 2}(\log 2)(\cos \pi x)(\pi)$$

$$= 2^{\sin \pi x}\, \pi \log 2 \cos \pi x.$$

H.W

3) S.T the fun. $y = e^{2x}(A\cos 3x + B\sin 3x)$ satisfies the differential eqn. $y'' - 4y' + 13y = 0$.

4) Find the derivative of (i) $y = \cos\left(\frac{1-e^{2x}}{1+e^{2x}}\right)$ (ii) $y = e^{\sqrt{x}}$.

DERIVATIVES OF IMPLICIT FUNCTIONS – DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS:

Formulae:

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

1) If $x^2+y^2=a^2$, find $\frac{dy}{dx}$. Also find an eqn. of the tangent to the circle at $(3,4)$.

Soln.- Gn.- $x^2+y^2=a^2$

$$2x+2y\frac{dy}{dx}=0 \Rightarrow x+y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow m = \left(\frac{dy}{dx}\right)_{(3,4)} = -\frac{3}{4}$$

An eqn. of the tangent is $(y-y_1)=m(x-x_1)$

$$\Rightarrow y = -\frac{3}{4}x+\frac{25}{4}.$$

2) If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then find $\frac{dy}{dx}$

Soln.- Gn.- $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

3) If $e^{x/y}=x-y$, then find $\frac{dy}{dx}$

Soln.- Gn.- $e^{x/y}=x-y$.

$$e^{x/y}\left[\frac{y(1)-x\frac{dy}{dx}}{y^2}\right] = 1-\frac{dy}{dx}$$

$$e^{x/y}\left[y-x\frac{dy}{dx}\right] = y^2-y^2\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2-ye^{x/y}}{y^2-xe^{x/y}}.$$

4) If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\cdots}}}$ Find $\frac{dy}{dx}$.

Soln.- Gn.- $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\cdots}}}$

$$\Rightarrow y=\sqrt{\sin x+y} \Rightarrow y^2=\sin x+y$$

$$\Rightarrow 2y\frac{dy}{dx} = \cos x+\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

5) If $e^y\cos x = 1+\sin(xy)$ find $\frac{dy}{dx}$

Soln.- Gn.- $e^y\cos x = 1+\sin(xy)$

$$e^y(-\sin x)+\cos x\, e^y\cdot\frac{dy}{dx} = 0+\cos xy\left[x\frac{dy}{dx}+y\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\cos(xy)+e^y\sin x}{e^y\cos x-x\cos(xy)}$$

6) Find $\frac{dy}{dx}$ if $\tan^{-1}(x^2y)=x+xy^2$

Soln.- Gn.- $\tan^{-1}(x^2y)=x+xy^2$

$$\frac{1}{1+(x^2y)^2}\left[x^2\frac{dy}{dx}+y\,2x\right] = 1+x\left[2y\frac{dy}{dx}\right]+y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+x^4y^2+y^2+x^4y^4-2xy}{x^2-2xy-2x^5y^3}.$$

7) If $x^3+y^3=16$ find the value of $\frac{d^2y}{dx^2}$ at $(2,2)$.

Soln.- Gn.- $x^3+y^3=16$

$\Rightarrow 3x^2+3y^2\frac{dy}{dx}=0$

$x^2+y^2\frac{dy}{dx}=0 \quad \Rightarrow \left(\frac{dy}{dx}\right)_{(2,2)}=-1$

$\Rightarrow 2x+y^2\frac{d^2y}{dx^2}+\frac{dy}{dx}2y\frac{dy}{dx}=0$

$2x+y^2\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^2=0 \quad \rightarrow (1)$

At $(2,2)$

$(1) \Rightarrow \frac{d^2y}{dx^2}=-2$

DERIVATIVES OF LOGARITHMIC FUNCTIONS:

Formulae:

1. $\frac{d}{dx}\log_a x = \frac{1}{x\log_e a}$
2. $\frac{d}{dx}\log_e x = 1/x$
3. $\frac{d}{dx}\log_e u = 1/u \cdot \frac{du}{dx}$ (or) $\frac{d}{dx}\log_e g(x) = \frac{1}{g(x)}g'(x)$
4. $\frac{d}{dx}\log_e|x| = 1/x$
5. $\frac{d}{dx}(a^b)=0$ [$\because$ a & b are constants]
6. $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$
7. $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}[\log_e a]\, g'(x)$
8. To find $\frac{d}{dx}[f(x)]^{g(x)}$, use logarithmic differentiation
9. $e = \lim_{x\to 0}(1+x)^{1/x}$
10. $e = \lim_{n\to\infty}(1+1/n)^n$.

1) Find $\frac{dy}{dx}$ if $y=x^{\sqrt{x}}$.

Soln.- Gn.- $y=x^{\sqrt{x}}$

$\log y = \sqrt{x}\log x$

$\frac{1}{y}\cdot\frac{dy}{dx} = \sqrt{x}(1/x)+\log x\left[\frac{1}{2\sqrt{x}}\right]$

$\frac{dy}{dx} = y\left[\frac{2+\log x}{2\sqrt{x}}\right] = x^{\sqrt{x}}\left[\frac{2+\log x}{2\sqrt{x}}\right]$

2) Find y' if $y=x^{\sin x}$

Soln.- $y=x^{\sin x}$

$\Rightarrow \log y = \sin x\log x$

$\frac{1}{y}\cdot\frac{dy}{dx} = \sin x(1/x)+\log x(\cos x)$

$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x}+\log x\cos x\right]$

3) Find y' if $y=(\sin x)^{\cos x}$

Soln.- Gn.- $y=(\sin x)^{\cos x}$

$\log y = \cos x\log\sin x$.

$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x\left[\frac{1}{\sin x}\cos x\right]+\log\sin x(-\sin x)$

$\frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x-\sin^2 x\log(\sin x)}{\sin x}\right]$

4) Find y' if $y=\log|x|$

Soln.- Gn.- $y=\log|x| = \begin{cases}\log x & , x>0\\ \log(-x) & , x<0\end{cases}$

$y' = \begin{cases}1/x & , x>0\\ -1/x(-1) & , x<0\end{cases} \Rightarrow y'=1/x \ , x\neq 0.$

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5) Find an eqn. of the tangent line to the curve at the gn. point $y = \log(x^2-3x+1)$ at $(3,0)$.

Soln. Gn. $y = \log(x^2-3x+1)$

$$\frac{dy}{dx} = \frac{1}{x^2-3x+1}(2x-3)$$

$$m = (y')_{(3,0)} = 3$$

Eqn. of tangent line is $y - y_1 = m(x-x_1)$

$$y = 3x-9.$$

H.W

6) If $y = (\cot x)^{\sin x} + (\tan x)^{\cos x}$, then find $\frac{dy}{dx}$.

7) Find y' if $y = (\sin x)^x$

8) Find y' if $y = x^{\sin x}$

9) Find y' if $y = x^x$

10) Find y' & y'' if $y = x^2 \log 2x$.

1) For what value of the constant 'c' is the fun. 'f' continuous on $(-\infty, \infty)$, $f(x) = \begin{cases} cx^2+2x & ; x<2 \\ x^3-cx & ; x\geq 2. \end{cases}$

Soln. Gn. $f(x) = \begin{cases} cx^2+2x & , x<2 \\ x^3-cx & , x\geq 2 \end{cases}$

At $x=2$

Gn. $f(x)$ is continuous on $(-\infty, \infty)$

$$f(2^-) = f(2) = f(2^+) \longrightarrow (A)$$

$f(2^-) = 4c+4$; $f(2) = 8-2c$

(A) => $\boxed{c = \frac{2}{3}}$

2) Does the curve $y = x^4-2x^2+2$ have any horizontal tangents? If so where?

Soln. Tangents are horizontal => $\frac{dy}{dx} = 0$.

Gn. $y = x^4 - 2x^2 + 2$

$$\Rightarrow x = 0, 1, -1$$

$\therefore$ The curve will have ~~horiz~~ horizontal tangents at $(0,2), (1,1), (-1,1)$.

DERIVATIVES OF HYPERBOLIC FUNCTIONS:

Formulae:

1) $\frac{d}{dx}(\sinh x) = \cosh x$

2) $\frac{d}{dx}(\cosh x) = \sinh x$

3) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

4) $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$

5) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

6) $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$.

7) $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

8) $\frac{d}{dx}(\operatorname{cosech}^{-1} x) = \frac{-1}{|x|\sqrt{x^2+1}}$

9) $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

10) $\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$

11) $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

12) $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

1) Find $\frac{dy}{dx}$ if $y = \operatorname{sech}^2(e^t)$.

$$\frac{dy}{dx} = -2\operatorname{sech}(e^t)\operatorname{sech}(e^t)\tanh(e^t)e^t$$

$$= -2\operatorname{sec}^2 h(e^t)\tanh(e^t)e^t.$$

2) Find $\frac{dy}{dx}$ if $y = x \sinh^{-1}(x/3) - \sqrt{9+x^2}$

Soln:-

Gn. $y = x \sinh^{-1}(x/3) - \sqrt{9+x^2}$

$$\frac{dy}{dx} = x \frac{1}{\sqrt{1+(x/3)^2}} \cdot (1/3) + \sinh^{-1}(x/3)(1) - \frac{1}{2\sqrt{9+x^2}}(2x)$$

$$= \sinh^{-1}(x/3).$$

3) Find $\frac{dy}{dx}$ if $y = \coth^{-1}(\sec x)$

Soln:- Gn. $y = \coth^{-1}(\sec x)$

$$\frac{dy}{dx} = \frac{1}{1-\sec^2 x} \sec x \tan x$$

$$= -\cot x \sec x = -\operatorname{cosec} x.$$

H.W

4) Find $\frac{dy}{dx}$ if $y = \tanh^{-1}(\sin x)$

5) Find $\frac{dy}{dx}$ if $y = x \sinh x - \cosh x$.

MAXIMA AND MINIMA OF FUNCTIONS OF ONE VARIABLE:

ABSOLUTE MAXIMUM AND ABSOLUTE MINIMUM:

Defn.: 1 Let c be a number in the domain D of a function f. Then $f(c)$ is the

* absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
* absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

Defn.: 2

The number $f(c)$ is a

* local maximum value of f if $f(c) \geq f(x)$ when x is near c.
* local minimum value of f if $f(c) \leq f(x)$ when x is near c.

THE EXTREME VALUE THEOREM:

If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$.

FERMAT'S THEOREM:

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Defn.: 3

A critical number of a fun. f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Defn.: 4

If f has a local maximum or minimum at c, then c is a critical number of f.

Increasing / Decreasing Test:

* If $f'(x) > 0$ on an interval, then f is increasing on that interval.
* If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test:

Suppose that c is a critical no. of a continuous funct.

a) If f' changes from positive to negative at c, then f has a local maximum at c.

b) If f' changes from negative to positive at c, then f has a local minimum at c.

c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Defn:

If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

NOTE: Concave upward $\equiv$ Convex downward

Concave downward $\equiv$ Convex upward

Concavity Test:

i) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

ii) If $f''(x) < 0$ for all x in I, then the graph f is concave downward on I.

Defn:

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test:

Suppose f'' is continuous near c.

i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c

ii) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

1) Find the critical values of the function:

(a) $g(x) = 2x^3 - 3x^2 - 36x$ (b) $g(y) = \frac{y-1}{y^2 - y + 1}$

(c) $f(\theta) = 2\cos\theta + \sin^2\theta$ (d) $f(x) = x^2 - 32\sqrt{x}$

(e) $f(x) = x^{3/4} - 2x^{1/4}$.

Soln.

a) Ans. $g(x) = 2x^3 - 3x^2 - 36x$

Critical nos. of g occur at $g'(x) = 0$

$$g'(x) = 6x^2 - 6x - 36$$

$\Rightarrow 6x^2 - 6x - 36 = 0$

$\Rightarrow x = -2, 3$

$\therefore$ critical value $= -2, 3$

b) Ans. $g(y) = \frac{y-1}{y^2 - y + 1}$

Critical nos. of g occur at $g'(y) = 0$.

$g'(y) = \frac{y(2-y)}{(y^2 - y + 1)^2} \Rightarrow \frac{y(2-y)}{(y^2-y+1)^2} = 0$

$\Rightarrow y(2-y) = 0 \Rightarrow y = 0, 2$

Critical value $= 0, 2$.

c) Given $f(\theta) = 2\cos\theta + \sin^2\theta$

Critical nos. of f occur at $f'(x) = 0$

$f'(\theta) = -2\sin\theta + 2\sin\theta\cos\theta$

$\Rightarrow -2\sin\theta\,[1-\cos\theta] = 0$

$\Rightarrow \theta = n\pi$, n is an integer.

$\therefore$ Critical value $= n\pi$, n is an integer.

d) Given $f(x) = x^2 - 32\sqrt{x}$

Critical nos. of f occur at $f'(x) = 0$.

$f'(x) = 2x - \frac{16}{\sqrt{x}}$

$\Rightarrow 2x - \frac{16}{\sqrt{x}} = 0 \Rightarrow x^3 = 64$

$\Rightarrow x = 4, -2 \pm i(3.464)$

$\therefore$ Critical value $= 4$ (real no. only).

e) Given $f(x) = x^{3/4} - 2x^{1/4}$

Critical nos. of f occur at $f'(x) = 0$.

$f'(x) = \frac{3}{4}x^{-1/4} - \frac{1}{2}x^{-3/4}$

$\Rightarrow \frac{3}{4}x^{-1/4} - \frac{1}{2}x^{-3/4} = 0$

$\Rightarrow x^{1/2} = \frac{2}{3} \Rightarrow x = \frac{4}{9}$

$\therefore f'(x)$ does not exist when $x = 0$.

$\therefore$ Critical value $= 0, \frac{4}{9}$

2) Find the absolute and local maximum and minimum values
of f.

a) $f(x) = \frac{1}{2}(3x-1)$, $x \le 3$

b) $f(x) = \log x$, $0 < x \le 2$

c) $f(x) = \begin{cases} 1-x, & 0 \le x < 2 \\ 2x-4, & 2 \le x \le 3 \end{cases}$

d) $f(x) = |x|$, $-1 < x < 2$

e) $f(x) = \sin x$, $0 \le x < \frac{\pi}{2}$.

Soln:

a) Given $f(x) = \frac{1}{2}(3x-1)$, $x \le 3$

x	...	1	2	3
$f(x)$	...	1	5/2	4

$\therefore f(3) = \frac{1}{2}(9-1) = 4$ is the absolute maximum of f

b) Given $f(x) = \log x$, $0 < x \le 2$

x	...	2
$f(x)$	...	$\log 2$

$f(2) = \log 2$ is the absolute maximum.

c) $f(x) = \begin{cases} 1-x, & 0 \le x < 2 \\ 2x-4, & 2 \le x \le 3. \end{cases}$

x	0	1	2	3
$f(x)$	1	0	0	2

$\therefore f(3) = 2$ is the absolute maximum

d) Given $f(x) = |x|$, $-1 < x < 2$

x	-1	0	1	2
$f(x)$	1	0	1	2

$\therefore f(0) = 0$ is the absolute minimum.

No, absolute maximum.

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e) Gn. $f(x) = \sin x$, $0 \le x < \pi/2$

x	0	...
$f(x)$	0	...

$f(0) = 0$, is the absolute minimum.

3) The closed interval method. Find the absolute maximum & minimum values of a) $f(x) = 3x^4 - 16x^3 + 18x^2$, $-1 \le x \le 4$
b) $f(x) = x e^{-x^2/8}$, $[-1, 4]$ (c) $f(x) = 2\cos x + \sin 2x$, $[0, \pi/2]$

Soln.

a) Gn. $f(x) = 3x^4 - 16x^3 + 18x^2$, $-1 \le x \le 4$

Critical nos. of f occur at $f'(x) = 0$. $\longrightarrow (1)$

$$f'(x) = 12x^3 - 48x^2 + 36x$$

$$(1) \Rightarrow 12x^3 - 48x^2 + 36x = 0$$

$$\Rightarrow x = 0, 3, 1$$

$\therefore$ Critical values are 0, 3, 1.

	End pt	critical pt.	C.P	C.P	End Point
x	-1	0	1	3	4
$f(x)$	37	0	5	-27	32

$f(3) = -27$ is the absolute minimum value of f

$f(-1) = 37$ is the absolute maximum value of f.

b) $x = \pm 2$

$\therefore$ Critical value $= 2$

$f(2) = 2e^{-1/2}$ is the absolute maxi. , $f(-1) = -e^{-1/8}$ is the absolute mini.

c) $f'(x) = -2\sin x + 2\cos 2x$

$\Rightarrow \cos 2x = \sin x$

$\Rightarrow x = \pi/6$

$f(\pi/6) = \frac{3}{2}\sqrt{3}$ is the A. maxi.

$f(\pi/2) = 0$ is the Absolute mini.

H.W

4) Use calculus to find the exact minimum and maximum values of the function $f(x) = x - 2\sin x$, $0 \le x \le 2\pi$.

5) Let $f(x) = (x-2)^{2/3}$ (a) Does $f'(2)$ exists?
b) S.T the only local extreme value of f occurs at $x = 2$.
c) Does the result in (b) contradict the extreme value thm.?

6) S.T 5 is a critical no. of the fun. $g(x) = 2 + (x-5)^3$ but does not have a local extreme value at 5.

7) P.T the fun. $f(x) = x^{101} + x^{51} + x + 1$ has neither a local max. nor a local mini.

Soln.

Gn. $f(x) = x^{101} + x^{51} + x + 1$

Critical values of f occur at $f'(x) = 0$

$$f'(x) = 101x^{100} + 51x^{50} + 1$$

$$f'(x) > 0$$

The $f'(x)$ is also a polynomial having all positive co-effi. and even power. Thus, $f'(x)$ is always +ve. Hence, $f(x)$ is always increasing. The concavity of the fun. will not change. So, it does not have a local minimum or local maximum.

8) Answer the following question about the functions whose derivatives are given:
a) what are the critical points of f?
b) on what interval is f increasing or decreasing.
c) At what pts., if any, does f assume local maximum and minimum values.
d) Find intervals of concavity and the inflection points.

i) $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$

ii) $f(x) = x^4 - 2x^2 + 3$

iii) $f(x) = x^2 - x - \log x$.

Soln:-

i) Gn:- $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

$f'(x) = 0 \Rightarrow \cos x = \sin x$

$\Rightarrow x = \pi/4, 5\pi/4$.

a) Critical values are $\pi/4, 5\pi/4$.

b)

Interval	Sign of f'	Behaviour of f
$0 < x < \pi/4$	+	increasing
$\pi/4 < x < 5\pi/4$	−	decreasing
$\frac{5\pi}{4} < x < 2\pi$	+	increasing

c) The first derivative test tells us there is a local.

i) Maximum at $\pi/4$, $f(\pi/4) = \sin \pi/4 + \cos \pi/4 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

ii) Minimum at $\frac{5\pi}{4}$, $f(\frac{5\pi}{4}) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} = -\sqrt{2}$.

d) $f''(x) = -(\sin x + \cos x)$

$f''(x) = 0 \Rightarrow \sin x = -\cos x$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

Interval	Sign of f''	Behaviour of f
$0 < x < 3\pi/4$	−	Concave down
$3\pi/4 < x < 7\pi/4$	+	Concave up
$7\pi/4 < x < 2\pi$	−	Concave down.

e) Inflection points are $(\frac{3\pi}{4}, 0), (\frac{7\pi}{4}, 0)$ Since $f(\frac{3\pi}{4}) = 0$, $f(\frac{7\pi}{4}) = 0$.

9) Find the local maximum and minimum values of f using both the first and second derivative test.

i) $y = x^4 - 4x^3$ ii) $y = x^5 - 5x + 3$.

Soln:-

i) Gn:- $y = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f'(x) = 0 \Rightarrow 4x^3 - 12x^2 = 0$

$x = 0, 3$

The critical points are 0, 3.

Interval	Sign of f'	Behaviour of f
$(-\infty, 0)$	−	decreasing
$(0, 3)$	−	decreasing
$(3, \infty)$	+	Increasing

First derivative test tells us that f does not have a local maximum or minimum at 0.

$f''(x) = 12x^2 - 24x$

$f''(x) = 0 \Rightarrow 12x^2 - 24x = 0$

$x = 0, 2$.

Interval	$f''(x)$	Behaviour of f.
$(-\infty, 0)$	+	Concave up
$(0, 2)$	−	Concave down
$(2, \infty)$	+	Concave up.

The point $(0, 0)$ is an inflection point concave upward to concave downward.

The point $(2, -16)$ is an inflection point concave downward to concave upward.

The second derivative test gives no information about the critical no. 0. Since $f'(0)=0$, $f''(0)=0$.

But first derivative test gives f does not have a local maximum or minimum at 0.

ii) Soln.

Ans. $y = f(x) = x^5 - 5x + 3$

$f'(x) = 5x^4 - 5$

$f'(x) = 0 \Rightarrow x = 1, -1$

$\therefore$ The critical points are $1, -1$.

Interval	Sign of f'	Behaviour of f
$-\infty < x < -1$	+	increasing
$-1 < x < 1$	−	decreasing
$1 < x < \infty$	+	increasing

First derivative test tells us that

(i) local maximum at $x = -1$

$f(-1) = -1 + 5 + 3 = 7$

Second derivative test tells us that

ii) local minimum at $x = 1$

$f(1) = -1$

$f''(x) = 20x^3$

$f''(x) = 0 \Rightarrow x = 0$

Interval	$f''(x)$	Behaviour of f
$(-\infty, 0)$	−	Concave down
$(0, \infty)$	+	Concave up.

$f'(1) = 0$, $f''(1) = 20$, $f(1) = -1$ is a local minium

$f'(-1) = 0$, $f''(-1) = -20$, $f(-1) = 7$ is a local maximum

H.W

10) If the fun. $f(x) = x^3 + ax^2 + bx$ has the local minimum value $-\frac{2}{9}\sqrt{3}$ at $x = \frac{1}{\sqrt{3}}$. What are the values of a and b?

Soln.

Ans. $f(x) = x^3 + ax^2 + bx$

$f'(x) = 3x^2 + 2ax + b$

$f'(x) = 0 \Rightarrow 3x^2 + 2ax + b = 0$

Ans. $x = \frac{1}{\sqrt{3}}$ is a critical point ; $x = -\frac{1}{\sqrt{3}}$ is also a critical pt.

[$\because$ irrational roots occurs in pairs]

$\therefore 3x^2 + 2ax + b = (x - \frac{1}{\sqrt{3}})(x + \frac{1}{\sqrt{3}}) = x^2 - \frac{1}{3}$

Equating the terms

$2a = 0$ $\qquad b = -\frac{1}{3}$

$\Rightarrow \boxed{a = 0}$

$\therefore f(x) = x^3 - \frac{1}{3}x$ //

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10) Find the cubic func. $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

Soln.

Given $f(x) = ax^3 + bx^2 + cx + d$.

$f'(x) = 3ax^2 + 2bx + c$

$f'(x) = 0 \Rightarrow 3ax^2 + 2bx + c = 0$

Given The critical points are $-2, 1$.

$\Rightarrow 3ax^2 + 2bx + c = (x+2)(x-1)$

$3ax^2 + 2bx + c = x^2 + x - 2$

Equating the co-effi.

$a = 1/3$, $b = 1/2$, $c = -2$

Given $f(-2) = 3$, $f(1) = 0$

$f(1) = 0$

$\Rightarrow a + b + c + d = 0 \quad \Rightarrow \quad d = 7/6$

$\therefore (1) \Rightarrow \quad f(x) = \frac{1}{6}\left[2x^3 + 3x^2 - 12x + 7\right]$

UNIT - IV

DIFFERENTIAL CALCULUS OF SEVERAL VARIABLES

LIMITS AND CONTINUITY:

Limits: The function $f(x,y)$ is said to tend to the limit l as $x \to a$ & $y \to b$, iff the limit l is independent of the path followed by the point (x,y) as $x \to a$ and $y \to b$.

Then $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = l$.

Continuity: A function $f(x,y)$ is said to be continuous at the point (a,b), if $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y)$ exist and $= f(a,b)$.

Evaluate: $\lim\limits_{\substack{x \to \infty \\ y \to 2}} \dfrac{xy+5}{x^2+2y^2}$

Soln.

$$\underset{\substack{x \to \infty \\ y \to 2}}{Lt} \frac{xy+5}{x^2+2y^2} = \lim_{x \to \infty} \left[\frac{2x+5}{x^2+8}\right] = \lim_{x \to \infty} \frac{2+5/x}{x+8/x} = 2/\infty = 0$$

PARTIAL DERIVATIVES:

1) If $u = (x-y)(y-z)(z-x)$ Show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

$$\frac{\partial u}{\partial x} = -(x-y)(y-z) + (y-z)(z-x); \quad \frac{\partial u}{\partial y} = (x-y)(z-x) - (y-z)(z-x)$$

$$\frac{\partial u}{\partial z} = (x-y)(y-z) - (x-y)(z-x)$$

2) If $u = x^y$ Show that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$

Soln.- Gn.- $u = x^y = e^{y \log x}$ $\quad [\because a^x = e^{x \log a}]$

$$\frac{\partial u}{\partial y} = e^{y \log x} \cdot \log x \quad ; \quad u_{xy} = x^{y-1}[1 + y \log x] \to (1)$$

$$u_x = e^{y \log x}(y/x) \quad ; \quad u_{yx} = x^{y-1}[1 + y \log x] \to (2)$$

3) If $x = r\cos\theta$; $y = r\sin\theta$, find (i) $\frac{\partial x}{\partial r}$ (ii) $\frac{\partial y}{\partial \theta}$

(iii) $\frac{\partial r}{\partial x}$ (iv) $\frac{\partial \theta}{\partial y}$

Soln.

$\frac{\partial x}{\partial r} = \cos\theta$; $\frac{\partial y}{\partial \theta} = r\cos\theta$; $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ $[\because r^2 = x^2+y^2$, $r = \sqrt{x^2+y^2}]$

$\frac{\partial \theta}{\partial y} = \frac{1}{1+y^2/x^2}\left(1/x\right) = \frac{x}{x^2+y^2}$ $[\because \theta = \tan^{-1} y/x]$

EULER THEOREM ON HOMOGENEOUS FUNCTION:

If u is a homogeneous function of degree n in x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

In general, if u be a homogeneous function of degree n in $x, y, z, t, \dots$ then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + t\frac{\partial u}{\partial t} + \dots = nu.$$

1) If $u = x/y + y/z + z/x$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Soln. Let $u(x,y,z) = x/y + y/z + z/x$

$u(tx,ty,tz) = t^0 u(x,y,z)$

$\Rightarrow u$ is a homogeneous function of x,y,z in degree 0

By Euler's thm. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

2) If $u = \sin^{-1}\left[\frac{x^3-y^3}{x+y}\right]$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$.

Soln.

Let $f(x,y) = \sin u = \frac{x^3-y^3}{x+y}$

$f(tx,ty) = t^2 f(x,y)$

Here $f = \sin u$

$\frac{\partial f}{\partial x} = \cos u \cdot \frac{\partial u}{\partial x}$; $\frac{\partial f}{\partial y} = \cos u \frac{\partial u}{\partial y}$

By Euler's thm.

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f \longrightarrow (1)$

(1) $\Rightarrow$ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$.

3) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Soln.

$f(x,y) = \tan u = \frac{x^3+y^3}{x-y}$

$f(tx,ty) = t^2 f(x,y)$. f is a homogeneous fun. of degree 2 in x & y

By Euler's thm. $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$

$\frac{\partial f}{\partial x} = \sec^2 u \frac{\partial u}{\partial x}$; $\frac{\partial f}{\partial y} = \sec^2 u \cdot \frac{\partial u}{\partial y}$.

H.W

4) If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$

5) If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

6) If $u = \frac{x^2+y^2}{\sqrt{x+y}}$, P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$

7) If $u = f\left(x/y, y/z, z/x\right)$, P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

1) If u is a homogeneous function of degree n in x & y,

S.T $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$.

Soln. By Euler's thm. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \longrightarrow (1)$

Diff. (1) p.w.r.to 'x', we get

$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = (n-1)\frac{\partial u}{\partial x} \longrightarrow (2)$

Diff. (1) p.w.r to 'y' we get

$y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y} \longrightarrow (3)$

(2) $\times x$ + (3) $\times y \Rightarrow$

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$.

2) If $u = (x-y)f(y/x)$ find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Soln.

$u(x,y) = (x-y)f(y/x)$

$u(tx,ty) = t^1 u(x,y)$

By Euler's thm.

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

H.W

3) If $u = \sin^{-1}\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ then $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$

TOTAL DERIVATIVES - DIFFERENTIATION OF IMPLICIT FUNCTIONS:

TOTAL DERIVATIVE: If $u = f(x,y)$, where $x = \phi(t)$ and $y = \psi(t)$, then we can express u as a function of t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$\dfrac{du}{dt}$ is called Total Derivative.

NOTE: Put $t = x$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

NOTE:

COMPOSITE FUNCTION OF ONE VARIABLE:

If $u = f(x,y,z)$ where x, y, z are all functions of a variable 't', then $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dt} + \dfrac{\partial u}{\partial z}\cdot\dfrac{dz}{dt}$

COMPOSITE FUNCTION OF TWO VARIABLES:

If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$, then z is a function of u, v.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

DIFFERENTIATION OF IMPLICIT FUNCTIONS:

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\partial f/\partial y} \qquad [\because \partial f/\partial y \neq 0]$$

1) Find $\dfrac{dy}{dx}$ when $x^3 + y^3 = 3axy$

Soln.- Let $f(x,y) = x^3 + y^3 - 3axy$

$$dy/dx = -\frac{x^2 - ay}{y^2 - ax}$$

2) If $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$, find $\dfrac{du}{dx}$

Soln.-

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 1 + (\log x + \log y) + x/y \cdot dy/dx \longrightarrow (1)$$

Gn.- $x^3 + y^3 + 3xy = 1$

Diff.- w.r.to 'x' we get

$$3x^2 + 3y^2\, dy/dx + 3[y + x\, dy/dx] = 0$$

$$dy/dx = \frac{-(y+x^2)}{x+y^2}$$

$$(1) \Rightarrow \frac{du}{dx} = \log x + \log y + 1 - \frac{x(y+x^2)}{y(x+y^2)}.$$

3) If $g(x,y) = \psi(u,v)$ where $u = x^2 - y^2$ and $v = 2xy$ Prove that

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right]$$

Soln.-

$$\frac{\partial g}{\partial x} = \frac{\partial \psi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \psi}{\partial v}\cdot\frac{\partial v}{\partial x} = 2x\frac{\partial \psi}{\partial u} + 2y\frac{\partial \psi}{\partial v}$$

$$\partial/\partial x = 2x\,\partial/\partial u + 2y\,\partial/\partial v$$

$$\frac{\partial g}{\partial y} = -2y\frac{\partial \psi}{\partial u} + 2x\frac{\partial \psi}{\partial v} \Rightarrow \partial/\partial y = -2y\,\partial/\partial u + 2x\,\partial/\partial v$$

$$\frac{\partial^2 g}{\partial x^2} = \partial/\partial x\left(\partial g/\partial x\right) \quad ; \quad \frac{\partial^2 g}{\partial y^2} = \partial/\partial y\left(\frac{\partial g}{\partial y}\right)$$

4) Given the transformations $u = e^x \cos y$ and $v = e^x \sin y$ and that ϕ is a function of u and v and also of x and y, P.T

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right).$$

Soln.

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x} \quad ; \quad \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

5) If $x^y + y^x = c$ find $\frac{dy}{dx}$

Soln.

Let $f(x,y) = x^y + y^x - C = 0$

$$\frac{dy}{dx} = -\frac{(\partial f/\partial x)}{(\partial f/\partial y)} = -\frac{y x^{y-1} + y^x \log y}{x^y \log x + x y^{x-1}}$$

6) If $z = f(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$. S.T

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

Soln.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{x}{r}\frac{\partial z}{\partial x} + \frac{y}{r}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} \qquad [x^2+y^2=r^2]$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left[\frac{x}{r}\frac{\partial z}{\partial x} + \frac{y}{r}\frac{\partial z}{\partial y}\right]^2 \; ; \; \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(-y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right)^2$$

7) If $u = \log(x^2+y^2) + \tan^{-1}(y/x)$. P.T $u_{xx} + u_{yy} = 0$

Soln. $u_x = \frac{2x-y}{x^2+y^2}$; $u_y = \frac{2y+x}{x^2+y^2}$ $\qquad \tan^{-1}(x) = \frac{1}{1+x^2}$ (1)

$$u_{xx} = \frac{2y^2-2x^2+2xy}{(x^2+y^2)^2} \; ; \; u_{yy} = \frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}.$$

H.W

1) If $F = f(x,y)$, $x = e^u \sin v$, $y = e^u \cos u$ S.T

$$\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = (x^2+y^2)\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) = e^{2u}\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right]$$

2) If $z = f(y-z, z-x, x-y)$, S.T $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial z} = 0$

3) If z is a fun of x & y and u & v are other two variables, Such that $u = lx+my$, $v = ly-mx$. S.T $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

2) $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$

JACOBIANS AND PROPERTIES:

Defn.: JACOBIAN

If $u_1, u_2, \ldots, u_n$ are functions of n variables $x_1, x_2, \ldots, x_n$, then the Jacobian of the transformation from $x_1, x_2, \ldots, x_n$ to $u_1, u_2, \ldots, u_n$ is defined by (1)

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} \longrightarrow (1)$$

$$\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

and is denoted by the symbol $\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots x_n)}$ or $J_{(u_1, u_2, \ldots u_n)}$

Property: 1

If u and v are the functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

[Inverse property of Jacobians]

Property : 2 - If u, v are functions of x, y and x, y are themselves functions of r, s, then $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(r,s)}$.

Property : 3 - If u, v, w are functionally dependent functions of three independent variables x, y, z, then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

1) If $x = r\cos\theta$, $y = r\sin\theta$, find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$

Soln.

(i) $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = r$

(ii) W.K.T $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1 \Rightarrow \frac{\partial(r,\theta)}{\partial(x,y)} = 1/r$

2) If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution.

Soln.

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \partial x/\partial\theta \\ \frac{\partial y}{\partial r} & \partial y/\partial\theta \end{vmatrix} = -4r^3$$

3) Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3. if $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$ (or) $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

Soln.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4.$$

4) Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Soln.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

H.W

1) If $u = \frac{x+y}{x-y}$ and $v = \tan^{-1}x + \tan^{-1}y$, find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$.

2) If $u = x-y$, $v = y-z$, $w = z-x$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

3) If $x = u(1+v)$ and $y = v(1+u)$, find $\frac{\partial(x,y)}{\partial(u,v)}$

4) If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$, find $\frac{\partial(u,v)}{\partial(x,y)}$

TAYLOR'S SERIES FOR FUNCTIONS OF TWO VARIABLES:

FORMULA: $f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots$$

where $x-a = h$, $y-b = k$

1) (i) Expand $e^x\cos y$ about $(0, \pi/2)$ [$(x-0), (y-\frac{\pi}{2})$] upto the third term using Taylor's series (ii) $e^x\cos y$ in powers of x & y as far as the terms of the third degree.

Soln.

Function	value at $(0,\pi/2)$	value at $(0,0)$

W.K.T

$$f(x,y) = f(a,b) + \frac{1}{1!}[hf_x(a,b) + kf_y(a,b)]$$
$$+ \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)]$$
$$+ \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)]$$
$$+ \cdots$$

$h = x-a$
$k = y-b$

(i) $a = 0, b = \pi/2$ (ii) $a = 0, b = 0$

2) Expand $e^x\log(1+y)$ in powers of x and y upto terms of third degree at $(0,0)$

$f_x = e^x\log(1+y)$

$f_y = e^x(1+y)^{-1}$

H.W
1) Obtain terms upto the third degree in the Taylor series expansion of $e^x \sin y$ around the point $[1, \pi/2]$.
2) Expand $f(x,y) = e^{xy}$ in Taylor series at $(1,1)$ upto second degree.

MAXIMA AND MINIMA FOR FUNCTIONS OF TWO VARIABLES:

SUFFICIENT CONDITIONS:

If $f_x(a,b) = 0$, $f_y(a,b) = 0$ and $f_{xx}(a,b) = A$, $f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$, then

(i) $f(a,b)$ is maximum value if $AC - B^2 > 0$ and $A < 0$ (or $B < 0$)

(ii) $f(a,b)$ is minimum value if $AC - B^2 > 0$ and $A > 0$ (or $B > 0$)

(iii) $f(a,b)$ is not an extreme (saddle) if $AC - B^2 < 0$

(iv) If $AC - B^2 = 0$, the test is inconclusive.

STATIONARY VALUE:

A function $f(x,y)$ is said to be stationary at (a,b) or $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

1) Find the extreme values of the function
$$f(x,y) = x^3 + y^3 - 3x - 12y + 20.$$

Soln:
$f_x(x,y) = 3x^2 - 3$; $f_y(x,y) = 3y^2 - 12$

$A = f_{xx}(x,y) = 6x$, $B = f_{xy}(x,y) = 0$, $C = f_{yy}(x,y) = 6y$

To find the stationary points

$f_x = 0$	$f_y = 0$
$3x^2 - 3 = 0$	$3y^2 - 12 = 0$
$x = \pm 1$	$y = \pm 2$

$\therefore$ The stationary points are $(1,2), (1,-2), (-1,2), (-1,-2)$

	$(1,2)$	$(1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$	$6 > 0$	$6 > 0$	$-6 < 0$	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$ $AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	min. pt	Saddle pt	Saddle pt	max. point

$\therefore$ Maximum value of $f(x,y)$ is
$$f(-1,-2) = 38$$

$\therefore$ Minimum value of $f(x,y)$ is $f(1,2) = 2$.

Table
Point $A = f_{xx}$ $B = f_{xy}$ $C = f_{yy}$ $AC - B^2$ nature Extreme value

2) Find the extreme values of $f(x,y) = x^3y^2(1 - x - y)$.

Soln:
$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\qquad A = f_{xx}(x,y) = 6xy^2 - 12x^2y^2 - 6xy^3$

$f_y = 2x^3y - 2x^4y - 3x^3y^2$ $\qquad B = 6x^2y - 8x^3y - 9x^2y^2$

$C = 2x^3 - 2x^4 - 6x^3y$

To find the stationary pts

$f_x = 0$	$f_y = 0$
$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$	$2x^3y - 2x^4y - 3x^3y^2 = 0$
$x^2y^2[3 - 4x - 3y] = 0$	$x^3y[2 - 2x - 3y] = 0$
$\Rightarrow x = 0, y = 0, 4x + 3y = 3$	$\Rightarrow x = 0, y = 0, 2x + 3y = 2$

$4x + 3y = 3 \longrightarrow (1)$ $\qquad 2x + 3y = 2 \longrightarrow (2)$

$\Rightarrow x = \frac{1}{2}$, $y = \frac{1}{3}$

$\therefore$ The stationary pts are $(0,0), (\frac{1}{2}, \frac{1}{3}), (0,1), (0, \frac{2}{3}), (\frac{3}{4}, 0)$ & $(1,0)$

$\because$ Put $x = 0$ in (1) $\Rightarrow y = 1$ (ie.) $(0,1)$

Put $x = 0$ in (2) $\Rightarrow y = \frac{2}{3}$ (ie.) $(0, \frac{2}{3})$

Put $y = 0$ in (1) $\Rightarrow x = \frac{3}{4}$ (ie.) $(\frac{3}{4}, 0)$

Put $y = 0$ in (2) $\Rightarrow x = 1$ (ie.) $(1,0)$

Let $A = 6xy^2 - 12x^2y^2 - 6xy^3$

$B = 6x^2y - 8x^2y - 9x^2y^2$

$C = 2x^3 - 2x^4 - 6x^3y$

	$(0,0)$	$(\frac{1}{2},\frac{1}{3})$	$(0,1)$	$(0,\frac{2}{3})$	$(\frac{3}{4},0)$	$(1,0)$
A	0	$-\frac{1}{9} < 0$	0	0	0	0
B	0	$-\frac{1}{12}$	0	0	0	0
$AC-B^2$	0	$\frac{1}{144} > 0$	0	0	0	0
Conclusion	Inconclusive	Max. Point	I. Con	I. Con.	I. Con.	I. Con.

$(\frac{1}{2},\frac{1}{3})$ is a maximum point.

$\therefore$ Max value of $f(x,y)$ is $f(\frac{1}{2},\frac{1}{3}) = \frac{1}{432}$

H.W

1) Examine $f(x,y) = x^3 + y^3 - 12x - 3y + 20$ for its extreme values.

2) Find the maximum and minimum values of $x^2 - xy + y^2 - 2x + y$.

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS:

we define a function

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

where λ is called Lagrange Multiplier which is independent of x, y, z.

-4-

1) A rectangular box open at the top, is to have a volume of 32 cc. Find the dimensions of the box, that requires the least material for its construction.

Soln. Let x, y, z be the length, breath & height of the box

Surface area $= xy + 2yz + 2zx$ $\rightarrow$ (A)

Volume $= xyz = 32$ $\rightarrow$ (B)

Let the auxiliary function F be

$F(x,y,z) = (xy + 2yz + 2zx) + \lambda(xyz - 32)$ $\rightarrow$ (1)

where λ is Lagrange multiplier.

$F_x = \frac{\partial F}{\partial x} = y + 2z + \lambda yz$	$F_y = x + 2z + \lambda zx$	$F_z = 2x + 2y + \lambda xy$.

when F is extremum

$F_x = 0$	$F_y = 0$	$F_z = 0$
$y + 2z + \lambda yz = 0$ $\Rightarrow y + 2z = -\lambda yz$ $\Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda \rightarrow (1)$	$x + 2z + \lambda zx = 0$ $\Rightarrow x + 2z = -\lambda zx$ $\Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda \rightarrow (2)$	$2x + 2y + \lambda xy = 0$ $\Rightarrow 2x + 2y = -\lambda xy$ $\Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda \rightarrow (3)$

From (1) & (2)

$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$

$\Rightarrow x = y$ $\rightarrow$ (4)

From (2) & (3)

$\Rightarrow \frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$

$\Rightarrow y = 2z$ $\rightarrow$ (5)

From (4) & (5), we get

$x = y = 2z$

(B) Volume $= xyz = 32$ $\Rightarrow (2z)(2z)z = 32$

$z^3 = 8 \Rightarrow z = 2$, $x = 4$, $y = 4$

Cost minimum when $x = 4, y = 4, z = 2$

$\therefore$ The dimension of the box are $4, 4, 2$.

2) Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square metre.

Soln:- Let x, y, z be the length, breath & height of the box

Surface area $= xy + 2yz + 2zx = 432 \quad \longrightarrow (A)$

Volume $= xyz$

Let the Auxiliary fun F be

$$F(x,y,z) = xyz + \lambda(xy + 2yz + 2zx - 432)$$

From (1) & (2) $\Rightarrow x = y \rightarrow (4)$

From (2) & (3) $\Rightarrow y = 2z \longrightarrow (5)$

From (4) & (5) $x = y = 2z \longrightarrow (6)$

(A) $\Rightarrow xy + 2yz + 2zx = 432$

$z^2 = 36 \Rightarrow z = 6$, $x = 12$, $y = 12$

The dimension of the box are 12, 12, 6.

Maximum Volume = 864 cubic metres.

3) Find the maximum and minimum value of $x^2 + y^2 + z^2$ subject to the Condition $x + y + z = 3a$.

Let the Auxiliary function F be

$$F(x,y,z) = (x^2+y^2+z^2) + \lambda(x+y+z-3a) \longrightarrow (1)$$

From (1), (2) & (3)

$x = y = z \longrightarrow (4)$

Given- $x + y + z = 3a$

$3x = 3a \Rightarrow x = a \Rightarrow y = a$, $z = a$

$\therefore (a, a, a)$ is the pt where minimum value occur.

$\therefore$ The minimum value is $a^2 + a^2 + a^2 = 3a^2$.

H.W

1) A rectangular box open at the top is to have a gas capacity K. Find the dimensions of the box requiring least material for its Construction.

2) Find the maximum volume of the largest rectangular parallelopiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Soln:-

Let a vertex of such parallelopiped by x, y, z

Then, all other vertices will be $(\pm x, \pm y, \pm z)$

The sides of the solid be $2x, 2y, 2z$.

Volume $= 8xyz$

$$\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \longrightarrow (A)$$

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$F_x = 0$ $\quad F_y = 0$ $\quad F_z = 0$

$\frac{x^2}{a^2} = -\frac{4xyz}{\lambda} \longrightarrow (1)$ $\quad \frac{y^2}{b^2} = -\frac{4xyz}{\lambda} \longrightarrow (2)$ $\quad \frac{z^2}{c^2} = -\frac{4xyz}{\lambda} \longrightarrow (3)$

From (1), (2) & (3)

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \longrightarrow (4)$$

Given- $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\Rightarrow x = a/\sqrt{3}$

(4) $\Rightarrow y = b/\sqrt{3}$, $z = c/\sqrt{3}$

The extremum point is $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$

Maximum Volume $= 8\frac{abc}{3\sqrt{3}}$ //

3) A thin closed rectangular box is to have one edge equal to twice the other and Constant volume $72\ m^3$. Find the least surface area of the box.

Soln:- Let $x, y, 2y$ be the length, breath & height of the box resp.

Surface area $= 2xy + 2y(2y) + 2(x)(2y)$

$= 6xy + 4y^2 \longrightarrow (A)$

Volume. $xyz = 72$

$xy(2y) = 72 \Rightarrow 2xy^2 = 72 \Rightarrow xy^2 = 36 \longrightarrow (B)$

$x = 4$, $y = 3$.

minimum Surface $= 108$.

UNIT - III
INTEGRAL CALCULUS

The area Problem:

The area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S is bounded by the graph of continuous function f (where $f(x) \geq 0$), the vertical lines $x = a$ and $x = b$, and the x-axis.

Defn.

The area A of the region S that lies under the graph of the continuous fun. f is the limit of the sum of the areas of approximately rectangles.

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

Example: Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

Soln.

a) Gn. $a = 0$ and $b = 2$. the width of a subinterval is

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}.$$

$$\Rightarrow x_1 = 2/n, \quad x_2 = 4/n, \quad x_3 = 6/n. \quad x_i = \frac{2i}{n} \text{ and } x_n = \frac{2n}{n}$$

$$\therefore R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

$$= e^{-x_1}\Delta x + e^{-x_2}\Delta x + \cdots + e^{-x_n}\Delta x$$

$$= e^{-2/n}\left(\tfrac{2}{n}\right) + e^{-4/n}\left(\tfrac{2}{n}\right) + \cdots + e^{-2n/n}\left(\tfrac{2}{n}\right)$$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \tfrac{2}{n}\left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \cdots + e^{-2n/n}\right) \quad \text{(by defn.)}$$

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}.$$

b) (i) Let $n=4$ the subintervals of equal with $\Delta x = 0.5$ are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$ & $[1.5, 2]$.

The midpoints of these subintervals are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$ & $x_4^* = 1.75$, and the sum of the areas of the four approximating rectangle is

$$M_4 = \sum_{i=1}^{4} f(x_i^*)\Delta x$$

$$= f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x.$$

$$= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5)$$

$$\boxed{A = 0.8557}$$

(ii) Let $n=10$ the subintervals are $[0, 0.2]$, $[0.2, 0.4]$, $\cdots$ $[1.8, 2]$ and the midpoints are $x_1^* = 0.1$, $x_2^* = 0.3$, $x_3^* = 0.5$, $\cdots$, $x_{10}^* = 1.9$. Thus

$$A = M_{10} = f(0.1)\Delta x + f(0.3)\Delta x + f(0.5)\Delta x + \cdots + f(1.9)\Delta x$$

$$= 0.2\left(e^{-0.1} + e^{-0.3} + e^{-0.5} + \cdots + e^{-1.9}\right) = 0.8632$$

So, $n=10$ estimate is better than the estimate with $n=4$.

The Definite Integral:

The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

NOTE: The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a Riemann Sum, $\Delta x = \frac{b-a}{n}$; $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals.

Theorem: 1

If f is continuous on $[a,b]$, or if f has only a finite no. of jump discontinuous, then f is integrable on $[a,b]$

ie., $\int_a^b f(x)\,dx$ exists.

Theorem: 2

If f is integrable on $[a,b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

NOTE:

1) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (2) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (3) $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

1) Evaluate $\int_0^3 (x^2 - 2x)\,dx$ by using Riemann sum by taking right end points as the sample points.

Soln:-

Take n subintervals, we have $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$x_0 = 0$, $x_1 = 3/n$, $x_2 = 6/n$, $\ldots$ $x_i = \frac{3i}{n}$.

Since we are using right end points.

$$\therefore \int_0^3 (x^2-2x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\tfrac{3i}{n}\right)\left(\tfrac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right] = \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \lim_{n\to\infty} \frac{18}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n\to\infty} \frac{18}{n^2}\,\frac{n(n+1)}{2}$$

$$= 0.$$

H.W

2) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right end points and $a = 0$, $b = 3$ and $n = 6$.

$\Rightarrow R_6 = -3.9375$

MIDPOINT RULE:

$$\int_a^b f(x)\,dx = \int_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x\left[f(\bar{x}_1) + \dots + f(\bar{x}_n)\right]$$

Where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = $ midpoint of $[x_{i-1}, x_i]$.

3) Use the Midpoint Rule with $n=5$ to approximate $\int_1^2 \frac{1}{x}\,dx$.

Soln:-

The end points of the five sub-intervals are

End Points 1, 1.2, 1.4, 1.6, 1.8, 2

Midpoints	1.1	1.3	1.5	1.7	1.9
$f(x) = 1/x$	$1/1.1$	$1/1.3$	$1/1.5$	$1/1.7$	$1/1.9$

$$\Delta x = \frac{b-a}{n} = \frac{1}{5}$$

$$\int_1^2 \frac{1}{x}\,dx = \Delta x\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\right]$$

$$\approx 0.691908$$

Since $f(x) = \frac{1}{x} > 0$ for $1 \le x \le 2$, the integral represents an area, and the approximation gn. by the Midpoint Rule is the sum of the areas of the rectangles.

THE FUNDAMENTAL THEOREM OF CALCULUS: Suppose f is continuous on $[a,b]$.

* If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$
* $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an anti-derivative of f, that is $F' = f$.

-2-

Find the derivative of the following functions:

i) $g(x) = \int_0^x \sqrt{1+t^2}\,dt$ (ii) $y(x) = \int_x^5 3t\sin t\,dt$ (iii) $h(x) = \int_1^{e^x} \log t\,dt$

iv) $f(x) = \int^{\tan x} \sqrt{t+\sqrt{t}}\,dt$ (v) $y = \int_{-1}^{x} \frac{t^2}{t^2+4}\,dt - \int_3^x \frac{t^2}{t^2+4}\,dt$.

vi) $g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du$

Soln:

i) Given $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

$\therefore g'(x) = \sqrt{1+x^2}$ [$\because f(t) = \sqrt{1+t^2}$ is continuous by FTC 1]

ii) Given $y(x) = \int_x^5 3t\sin t\,dt = -\int_5^x 3t\sin t\,dt$

Put $t = x$
$dt = dx$.

$\therefore y'(x) = -3x\sin x$.

iii) Given $h(x) = \int_1^{e^x} \log t\,dt$

$u = e^x$

$\frac{dh}{dx} = \frac{dh}{du}\cdot\frac{du}{dx}$

$h'(x) = \log e^x\,(e^x)$

$= x e^x$

iv) Given $f(x) = \int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt$

$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$

$f'(x) = \sqrt{\tan x + \sqrt{\tan x}}\;d(\tan x)$

$= \left[\sqrt{\tan x + \sqrt{\tan x}}\right]\sec^2 x$.

v) Given $y = \int_{-1}^{x} \frac{t^2}{t^2+4}\,dt - \int_3^x \frac{t^2}{t^2+4}\,dt$

$y' = \frac{x^2}{x^2+4} - \frac{x^2}{x^2+4} = 0$

vi) Gn. $g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\, du$

$= -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\, du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\, du = I_1 + I_2 \longrightarrow (1)$

For I_1

Put $t = 2x$, $dt = 2dx$

$\frac{dg}{dx} = \frac{dg}{dt} \cdot \frac{dt}{dx}$

$I_1 = -2\left[\frac{4x^2-1}{4x^2+1}\right]$

For I_2

$I_2 = 3\left[\frac{9x^2-1}{9x^2+1}\right]$

$(1) \Rightarrow g'(x) = -2\left[\frac{4x^2-1}{4x^2+1}\right] + 3\left[\frac{9x^2-1}{9x^2+1}\right]$

Find the derivative of the following functions

1) Find the area under the cosine curve from 0 to b, where $0 \le b \le \pi/2$

Soln.

Gn. $f(x) = \cos x$.

Antiderivative $f(x) = \sin x$

$A = \int_0^b \cos x\, dx = F(b) - F(0)$ [by FTC 2]

$= \sin b - \sin 0 = \sin b$

2) $\int_{-1}^{2} (x^3 - 2x)\, dx$

Soln. Gn. $\int_{-1}^{2} (x^3-2x)\, dx$ Here $f(x) = x^3 - 2x$.

F = Antiderivative $f(x) = \frac{x^4}{4} - x^2$.

$A = \int_{-1}^{2} (x^3-2x)\, dx = F(2) - F(-1) = 3/4$.

3) $\int_0^{\pi} f(x)\, dx$ where $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x < \pi \end{cases}$

Soln.

Gn. $\int_0^{\pi} f(x)\, dx$ where $f(x) = \begin{cases} \sin x, & 0 \le x < \pi/2 \\ \cos x, & \pi/2 \le x < \pi \end{cases}$

F = Antiderivative $f(x) = \begin{cases} -\cos x, & 0 \le x < \pi/2 \\ \sin x, & \pi/2 \le x \le \pi \end{cases}$

$A = F_1(\pi/2) - F_1(0) + F_2(\pi) - F_2(\pi/2)$

$= 0$.

4) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\, dx$.

Soln. Here $f(x) = \frac{8}{1+x^2}$

Antiderivative $f(x) = 8\tan^{-1} x$.

$A = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\, dx = F(\sqrt{3}) - F(1/\sqrt{3})$

$= 8(\pi/3) - 8(\pi/6) = \frac{4}{3}\pi$

5) What is wrong with the equation?

$\int_{\pi/3}^{\pi} \sec\theta \tan\theta\, d\theta = [\sec\theta]_{\pi/3}^{\pi} = -3$.

Soln.

The fun. $f(\theta) = \sec\theta \tan\theta$ is not continuous on the interval $[\pi/3, \pi]$. So FTC 2 cannot be applied. $[\because \tan \pi/2 = \infty]$

THE INDEFINITE INTEGRALS:

Formulae:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ 2. $\int \frac{1}{x} dx = \log x + C$

3. $\int e^x dx = e^x + C$ 4. $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$.

Evaluate:

i) $\int \left(\frac{6}{x^2} + \sqrt{x} + x^{3/2} + \frac{5}{x} + 1\right) dx$ (ii) $\int (x^{2/5} - x^{-3/5})^2 dx$

iii) $\int \frac{ax^{-2} + bx^{-1} + k}{x^{-4}} dx$. (iv) $\int (e^{\log x} + 2) dx$

Soln.

i) $\Rightarrow 6\frac{x^{-1}}{-1} + \frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} + 5\log x + x + C$

ii) $\Rightarrow \frac{5}{9}x^{9/5} - 5x^{-1/5} - \frac{5}{2}x^{4/5} + C$

iii) $\Rightarrow \frac{a}{3}x^3 + \frac{b}{4}x^4 + \frac{k}{5}x^5 + C$

iv) $\Rightarrow \frac{x^2}{2} + 2x + C$.

Formulae: Pg. No. : 3.18

Evaluate (i) $\int \frac{1}{\sin^2 x \cos^2 x} dx$ (ii) $\int \frac{1}{1+\sin x} dx$

(iii) $\int \sqrt{1+\sin 2x}\, dx$ (iv) $\int (\tan x - 2\cot x)^2 dx$

(v) $\int \left[\cosh x + \frac{1}{x\sqrt{x^2-1}} + \frac{1}{1+x^2}\right] dx$

Soln.

(i) $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \tan x - \cot x + C$.

ii) Given. $\int \frac{1}{1+\sin x} dx = \int \frac{1}{1+\sin x} \cdot \frac{1-\sin x}{1-\sin x} dx$

$= \int \frac{1-\sin x}{\cos^2 x} dx = \tan x - \sec x + C$

iii) $\int \sqrt{1+\sin 2x}\, dx = \int \sqrt{(\sin x + \cos x)^2}\, dx$

$= -\cos x + \sin x + C$.

iv) $\int (\tan x - 2\cot x)^2 dx = \int (\tan^2 x + 4\cot^2 x - 4\tan x \cot x) dx$

$= \int \left[\sec^2 x - 1 + 4(\operatorname{cosec}^2 x - 1) - 4\tan x \cdot \frac{1}{\tan x}\right] dx$

$= \tan x - 4\cot x - 9x + C$.

v) $\int \left[\cosh x + \frac{1}{x\sqrt{x^2-1}} + \frac{1}{1+x^2}\right] dx$

$= \sinh x + \sec^{-1} x + \tan^{-1} x + C$.

Formulae: Pg. No: 3.22.

Max-min inequality

If f has maximum value $\max f$ and minimum value $\min f$ on $[a,b]$, then $(\min f)(b-a) \le \int_a^b f(x)\,dx \le (\max f)(b-a)$

Domination:

$f(x) \ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

$f(x) \ge 0$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge 0$.

1) Evaluate: (i) If $\int_0^{10} f(x)\,dx = 17$, $\int_0^{8} f(x)\,dx = 12$ then find $\int_8^{10} f(x)\,dx$.

(ii) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ (or) $\int_0^{\pi/2} \frac{1}{1+\cot x} dx$ (iii) $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$

iv) $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$. (v) $\int_0^{\pi/2} \log(\tan x)\,dx$.

Soln:-

(i) Gn:- $\int_0^{10} f(x)\,dx = 17$ & $\int_0^8 f(x)\,dx = 12$

To find: $\int_8^{10} f(x)\,dx$.

$$\int_0^{10} f(x)\,dx = \int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx.$$

$$\Rightarrow \int_8^{10} f(x)\,dx = 5.$$

(ii) Soln.

Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx \longrightarrow (1)$

$\Rightarrow I = \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\,dx$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right]$

$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx \longrightarrow (2)$

$(1)+(2) \Rightarrow 2I = \int_0^{\pi/2} dx = \pi/2$

$$\boxed{I = \pi/4}$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

(iii) $2I = \int_0^{\pi/2} dx$

$I = \pi/4$

(iv) $2I = \int_0^a \left[\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\right]dx = a$

(v) $2I = \int_0^{\pi/2} \log(\tan x \cdot \cot x)\,dx$

$I = 0.$

2) Evaluate i) $\int_{-1}^{1} \log\left(\frac{4-x}{4+x}\right)dx$ ii) $\int_{-\pi/2}^{\pi/2} \sin^{199} x\,dx$

Soln.

Let $I = \int_{-1}^{1} \log\left(\frac{4-x}{4+x}\right)dx$

Here, $f(x) = \log\left(\frac{4-x}{4+x}\right)$

$f(-x) = -f(x)$

$\Rightarrow f(x)$ is an odd fun.

$\therefore \int_{-1}^{1} \log\left(\frac{4-x}{4+x}\right)dx = 0$

$\Rightarrow f(x) = \sin^{199} x$

$\Rightarrow f(x)$ is an odd fun.

$\therefore \int_{-\pi/2}^{\pi/2} \sin^{199} x\,dx = 0$

3) Evaluate: $\int_0^{\pi} \sin^2 x \cos^3 x\,dx$.

Soln:

Let $I = \int_0^{\pi} \sin^2 x \cos^3 x\,dx$

Here, $f(x) = \sin^2 x \cos^3 x$

$\Rightarrow f(\pi - x) = \sin^2(\pi - x)\cos^3(\pi - x) = \sin^2 x(-\cos x)^3$

$= -\sin^2 x \cos^3 x = -f(x)$

$\therefore \int_0^{\pi} \sin^2 x \cos^3 x\,dx = 0$ $\left[\because \int_0^{2a} f(x)dx = 0 \text{ if } f(2a-x) = -f(x);\right.$ Here $\left.2a = \pi\right]$

4) Evaluate $\int_0^1 e^{-x^2}\,dx$

Soln:

Here, $f(x) = e^{-x^2}$ is a decreasing fun on $[0,1]$ and its absolute maximum is $M = f(0) = 1$ and its absolute min. is $m = f(1) = e^{-1}$.

Thus the property.

If $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$e^{-1}(1-0) \le \int_0^1 e^{-x^2}\,dx \le 1(1-0)$$

$$e^{-1} \le \int_0^1 e^{-x^2}\,dx \le 1$$

$$\Rightarrow 0.367 \le \int_0^1 e^{-x^2}\,dx \le 1.$$

5) S.T the value of $\int_0^1 \sqrt{1+\cos x}\,dx$ is less than or equal to $\sqrt{2}$.

Soln:-

The Max-Min inequality for definite integrals says that $\min f.(b-a)$ is a lower bound for the value of $\int_a^b f(x)\,dx$ and that $\max f.(b-a)$ is an upper bound.

The maximum value of $\sqrt{1+\cos x}$ on $[0,1]$ is $\sqrt{1+1} = \sqrt{2}$,

So.

$$\int_0^1 \sqrt{1+\cos x}\,dx \le \sqrt{2}\,.\,(1-0) = \sqrt{2}.$$

Defn:-

If f is integrable on $[a,b]$, then its average value on $[a,b]$, also called its mean, is

$$\text{avg}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

6) Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2,2]$

Soln:

The average value of f is

$$\text{avg}(f) = \frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}\,dx = \frac{\pi}{2}.$$

SUBSTITUTION RULE:

Integration by Substitution (or) Change of the independent Variable:

The method of substitution depends on finding a suitable Substitution to convert the gn. integral into a standard form.

TYPE 1: $\int [f(x)]^n f'(x)\,dx$ (or) $\int \phi[f(x)]\,f'(x)\,dx$

Substitute $u = f(x)$ and then proceed.

1) Evaluate the following

i) $\int \frac{1}{(ax+b)^4}\,dx$ (ii) $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$ (iii) $\int \frac{(1+\sqrt{x})^n}{\sqrt{x}}\,dx$

iv) $\int \frac{x^2}{(a+bx)^3}\,dx$ (v) $\int \frac{x^{n-1}}{a+bx^n}\,dx$. (vi) $\int \frac{\sec^2(\log x)}{x}\,dx$

Soln:

i) Let $u = ax+b \Rightarrow \frac{du}{dx} = a \Rightarrow dx = \frac{1}{a}du$

$$\Rightarrow \int \frac{1}{(ax+b)^4}\,dx = \int \frac{1}{u^4}\cdot\frac{1}{a}\,du = \frac{1}{a}\int u^{-4}\,du$$

$$= \frac{1}{a}\left(\frac{u^{-3}}{-3}\right)+C = -\frac{1}{3a}\,\frac{1}{(ax+b)^3}+C.$$

ii) Let $u = 2-\frac{1}{x} \Rightarrow du = \frac{1}{x^2}\,dx$

$$I = \frac{2}{3}\left(2-\frac{1}{x}\right)^{3/2}+C$$

iii) Let $u = (1+\sqrt{x}) \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$

$$I = \frac{2}{n+1}(1+\sqrt{x})^{n+1}+C$$

iv) Let $u = a+bx \Rightarrow du = b\,dx \Rightarrow I = \frac{1}{b^3}\left[\log(a+bx)+\frac{2a}{a+bx} - \frac{a^2}{2(a+bx)^2}\right]+C$

$x = \frac{u-a}{b}$

v) Let $u = a+bx^n \Rightarrow du = bn\,x^{n-1}\,dx.$

$$I = \frac{1}{nb}\log(a+bx^n)+C$$

vi) Let $I = \int \frac{\sec^2(\log x)}{x}\,dx$

Let $u = \log x$; $du = \frac{1}{x}\,dx$

$I = \int \sec^2 u\,du = \tan u + C = \tan(\log x) + C$

2) Evaluate (i) $\int e^{x^3} x^2\,dx$ (ii) $\int e^{\cos x} \sin x\,dx$ (iii) $\int (\log a)^x\,dx$

(iv) $\int x^3 \cos(x^4+2)\,dx$ (v) $\int e^{\tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right]dx$.

Soln:-

(i) Let $I = \int e^{x^3} x^2\,dx$

Put $u = e^{x^3}$; $du = e^{x^3}\,3x^2\,dx$

$I = \frac{1}{3} e^{x^3} + C$

(ii) Put $u = e^{\cos x}$

$I = -e^{\cos x} + C$

iii) Let $I = \int (\log a)^x\,dx$

$= \int e^{\log(\log a)^x}\,dx$

$= \frac{(\log a)^x}{\log(\log a)} + C$

(iv) Put $u = x^4 + 2$

$du = 4x^3\,dx$

$I = \frac{1}{4}\sin(x^4+2) + C$

v) Let $I = \int e^{\tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right]dx$

Put $u = \tan^{-1}x$; $du = \frac{1}{1+x^2}\,dx$

$\tan u = x$; $1+x+x^2 = 1+\tan u+\tan^2 u = \tan u + \sec^2 u$

$I = \int e^u[\tan u + \sec^2 u]\,du$

Put $t = e^u \tan u$; $dt = [e^u \sec^2 u + \tan u\, e^u]\,du$

$\therefore I = \int dt = t + C$

$= e^u \tan u + C = x\,e^{\tan^{-1}x} + C$

INTEGRATION BY PARTS:

$\int u\,dv = uv - \int v\,du$.

Bernoulli's formula:

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

Evaluate (i) $\int t\sec^2 2t\,dt$ (ii) $\int_0^{1/2} x\cos\pi x\,dx$ (iii) $\int (\log x)^2\,dx$

(iv) $\int \frac{x}{\sec x + 1}\,dx$ (v) $\int (x^2+2x)\cos x\,dx$ (vi) $\int e^{ax}\sin bx\,dx$

(vii) $\int \tan^{-1}x\,dx$. Also find $\int_0^1 \tan^{-1}x\,dx$

Soln:-

i) Gn:- $\int t\sec^2 2t\,dt$

Let $u = t$ $dv = \sec^2 2t\,dt$

$du = dt$ $v = \int \sec^2 2t\,dt = \frac{\tan 2t}{2}$

$\int u\,dv = uv - \int v\,du$

$\therefore \int t\sec^2 2t\,dt = \frac{1}{2}t\tan 2t - \frac{1}{4}\log(\sec 2t) + C$

ii) Let $u = x$ $dv = \cos\pi x\,dx$

$\int_0^{1/2} x\cos\pi x\,dx = \frac{\pi-2}{2\pi^2}$.

iii) $\int (\log x)^2\,dx$

Let $u = (\log x)^2$; $dv = dx$ $\Rightarrow \int (\log x)^2\,dx = x(\log x)^2 - 2\int \log x\,dx$

$du = 2\log x\,(\frac{1}{x})\,dx$; $v = x$

Take $u = \log x$; $dv = dx$ $\Rightarrow \int \log x\,dx = x\log x - x$

$\Rightarrow \int (\log x)^2\,dx = x(\log x)^2 - 2[x\log x - x] + C$

iv) $\int \frac{x}{\sec x+1}\,dx = \int \frac{x\cos x}{1+\cos x}\,dx$

$= \int \left[x - \frac{x}{1+\cos x}\right]dx = \int x\,dx - \int \frac{x}{1+\cos x}\,dx$

$\left[\because 1+\cos x = 2\cos^2 \frac{x}{2}\right]$

$= \int x\,dx - \int \frac{x}{2}\cdot\frac{1}{\cos^2 \frac{x}{2}}\,dx$

$= \int x\,dx - \frac{1}{2}\int x\sec^2 \frac{x}{2}\,dx$

Let $u = x$, $dv = \sec^2 \frac{x}{2}\,dx$

$v = 2\tan\frac{x}{2}$

$= \frac{x^2}{2} - x\tan\frac{x}{2} + 2\log\left[\sec\frac{x}{2}\right] + C$.

v) $\int (x^2+2x)\cos x\,dx$

Bernoulli's formula

Let $u = x^2+2x$ $\quad v = \cos x$

$\int (x^2+2x)\cos x\,dx = (x^2+2x-2)\sin x + (2x+2)\cos x + C$,

vi) Let $u = \tan^{-1}x$ $\quad dv = dx \Rightarrow v = x$

$du = \frac{1}{1+x^2}\,dx$

$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx \longrightarrow (1)$

Put $t = 1+x^2$, $dt = 2x\,dx$

$\Rightarrow \int \frac{x}{1+x^2}\,dx = \frac{1}{2}\log(1+x^2)$

$(1) \Rightarrow \int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C \longrightarrow (2)$

To find: $\int_0^1 \tan^{-1}x\,dx$

$= \frac{\pi}{4} - \frac{1}{2}\log 2$

REDUCTION FORMULA:

* $\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$, $n \geq 2$

$I_n = \frac{1}{n}[-\cos x\sin^{n-1}x] + \frac{n-1}{n} I_{n-2}$

* $\int_0^{\pi/2} \sin^n x\,dx$

$\Rightarrow I_n = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}\cdot 1 & \text{if } n \text{ is odd} \end{cases}$

* $\int \cos^n x\,dx = \frac{1}{n}[\cos^{n-1}x\sin x] + \frac{n-1}{n}\int \cos^{n-2}x\,dx$, $n \geq 2$.

* $\int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}\cdot 1 & \text{if } n \text{ is odd} \end{cases}$

* $\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1} I_{n-2}$, $n \geq 2$

* $\int \operatorname{cosec}^n x\,dx = \frac{-1}{n-1}\operatorname{cosec}^{n-2}x\cot x + \frac{n-2}{n-1} I_{n-2}$, $n \geq 2$

* $\int x^n e^x\,dx = x^n e^x - n I_{n-1}$, $n \geq 1$

* $\int (\log x)^n\,dx = x(\log x)^n - n I_{n-1}$

* $\int x^n \sin mx\,dx = \frac{-x^n\cos mx}{m} + \frac{n}{m^2}x^{n-1}\sin mx - \frac{n(n-1)}{m^2} I_{n-2}$

* $\int x^n \cos mx\,dx = \frac{x^n\sin mx}{m} + \frac{n}{m^2}x^{n-1}\cos mx - \frac{n(n-1)}{m^2} I_{n-2}$

* $\int \tan^n x\,dx$, $n \neq 1$ $= \frac{\tan^{n-1}x}{n-1} - I_{n-2}$

* $\int \cot^n x\,dx = \frac{-\cot^{n-1}x}{n-1} - I_{n-2}$

1) Using the reduction formulae, Find the values of (i) $\int \sin^4 x \, dx$ (ii) $\int \sin^3 x \, dx$.

Soln.

i) Gn. $\int \sin^4 x \, dx$, Here $n = 4$

$\Rightarrow I_4 = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x \, dx$ (use reduction formula)

$= -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4}\left[-\frac{\sin 2x}{4} + \frac{x}{2}\right] + C$

$= -\frac{1}{4} \cos x \sin^3 x - \frac{3}{16} \sin 2x + \frac{3}{8} x + C$

ii) $I_3 = -\frac{1}{3} \cos x \sin^2 x + \frac{2}{3} \int \sin x \, dx$

$= -\frac{1}{3} \cos x \sin^2 x - \frac{2}{3} \cos x + C$

2) Find the values of (i) $\int_0^{\pi/2} \sin^{2n+1} x \, dx$ (ii) $\int_0^{\pi/2} \sin^8 x \, dx$

Soln.

i) $\int_0^{\pi/2} \sin^{2n+1} x \, dx$, Here $n = 2n+1$ is odd.

$I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \ldots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$

ii) $\int_0^{\pi/2} \sin^8 x \, dx$, Here $n = 8$ is even

$I_8 = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

3) Find the values of i) $\int \cos^4 x \, dx$ (ii) $\int_0^{\pi/2} \cos^{10} x \, dx$

Soln.

i) $\int \cos^4 x \, dx$, Here $n = 4$

$I_4 = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx$

$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{16} \sin 2x + \frac{3}{8} x + C$

ii) $\int_0^{\pi/2} \cos^{10} x \, dx$, Here $n = 10$ is an even

$I_{10} = \frac{63}{512} \pi$

4) Evaluate i) $\int \cos \sqrt{x} \, dx$ (ii) $\int t^3 e^{-t^2} dt$ (iii) $\int x \log(1+x) \, dx$

Soln.

i) $\int \cos \sqrt{x} \, dx$

Put $t = \sqrt{x}$ $\Rightarrow dt = \frac{1}{2\sqrt{x}} dx$ $\Rightarrow 2t \, dt = dx$

$\Rightarrow \int \cos \sqrt{x} \, dx = \int \cos t (2t \, dt) = 2 \int t \cos t \, dt$

$= 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C.$

ii) $\int t^3 e^{-t^2} dt \longrightarrow (1)$

Put $x = e^{-t^2}$ $\Rightarrow dx = e^{-t^2}(-2t) \, dt$

$-\log x = t^2$ $\Rightarrow -\frac{dx}{2} = t e^{-t^2} dt$

$\log(1/x) = t^2$

$(1) \Rightarrow \int t^3 e^{-t^2} dt = \int \log(1/x) \left(-\frac{dx}{2}\right) = -\frac{1}{2} \int \log(1/x) \, dx$

$= \frac{1}{2} \int \log x \, dx \longrightarrow (2)$

put $u = \log x$ $\quad dv = dx$

$du = \frac{1}{x} dx$ $\quad v = x$

$\int \log x \, dx = (\log x)(x) - \int x \frac{1}{x} dx$

$= x \log x - x$

$(2) \Rightarrow \int t^3 e^{-t^2} dt = -\frac{1}{2} e^{-t^2} [1 + t^2] + C$

iii) $\int x \log(1+x) \, dx \longrightarrow (1)$

Put $t = 1 + x$ $\Rightarrow dt = dx$

$(1) \Rightarrow \int x \log(1+x) \, dx = \int (t-1) \log t \, dt \longrightarrow (2)$

Put $u = \log t$ $\quad dv = (t-1)\,dt$

$du = \frac{1}{t}\,dt$ $\quad v = \frac{t^2}{2} - t$

$$(2) \Rightarrow \int x \log(1+x)\,dx = (\log t)\left(\frac{t^2}{2}-t\right) - \int \left(\frac{t^2}{2}-t\right)\frac{1}{t}\,dt$$

$$= (\log t)\left(\frac{t^2}{2}-t\right) - \frac{1}{4}t^2 + t + C$$

$$= \log(1+x)\left[\frac{(1+x)^2}{2} - (1+x)\right] - \frac{1}{4}(1+x)^2 + (1+x) + C,$$

TRIGONOMETRIC INTEGRALS:

* $\int \sin^m x \cos^n x\,dx$

Case (i): If n is odd $(n = 2k+1)$, then

$$\int \sin^m x \cos^{2k+1} x\,dx = \int \sin^m x\,(\cos^2 x)^k \cos x\,dx$$

$$= \int \sin^m x\,(1-\sin^2 x)^k \cos x\,dx$$

Here, subs - $u = \sin x$

Case (ii): If m is odd $(m = 2k+1)$, then

$$\int \sin^{2k+1} x \cos^n x\,dx = \int (\sin^2 x)^k \cos^n x \sin x\,dx$$

$$= \int (1-\cos^2 x)^k \cos^n x \sin x\,dx$$

Here, subs - $u = \cos x$

NOTE: If both m and n are odd apply case (i) or case (ii)

Case (iii): If both m and n are even, use half-angle identities

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$; $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

* $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{2}{3+n}\cdot\frac{1}{1+n}$

(if m is odd, n may be even or odd)

$$= \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{1}{2+n}\cdot\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}\cdot 1$$

(if m is even, n is odd)

$$= \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{1}{2+n}\cdot\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

(if m is even, n is even)

* $\int \tan^m x \sec^n x\,dx$

Case (i) If m is odd $(m = 2k+1)$, then

$$\int \tan^{2k+1} x \sec^n x\,dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x\,dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\,dx$$

Subs - $u = \sec x$

Case (ii): If n is even $(n = 2k)$, then

$$\int \tan^m x \sec^{2k} x\,dx = \int \tan^m x\,(\sec^2 x)^{k-1} \sec^2 x\,dx$$

$$= \int \tan^m x\,(1+\tan^2 x)^{k-1} \sec^2 x\,dx$$

Subs - $u = \tan x$

Formulae:

$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$

$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$

$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$

1) Evaluate (i) $\int \sin^6 x \cos^3 x\, dx$ (ii) $\int_0^{\pi/2} \sin^7 y\, dy$

(iii) $\int_0^{\pi/3} \tan^5 x \sec^4 x\, dx$ (iv) $\int \cos^2 x \tan^3 x\, dx$ (v) $\int \tan^4 x\, dx$

(vi) $\int \sin 4x \cos 5x\, dx$ (vii) $\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$

Soln:-

i) Let $u = \sin x$; $du = \cos x\, dx$

$$I = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C$$

ii) $I = 16/35$

iii) $\int_0^{\pi/3} \tan^4 x \sec^4 x \tan x\, dx$ $\quad x \to 0 \Rightarrow u \to 1$

Put $u = \sec x$; $du = \sec x \tan x\, dx$ $\quad x \to \pi/3 \Rightarrow u \to 2.$

$$I = \int_1^2 (u^2-1)^2 u^3\, du = \frac{117}{8}$$

iv) $\int \cos^2 x \tan^3 x\, dx = \int \cos^2 x \tan^2 x \tan x\, dx$

$= \int \cos^2 x (\sec^2 x - 1) \tan x\, dx$

$= \int \cos^2 x \sec^2 x \tan x\, dx - \int \cos^2 x \tan x\, dx$

$= \int \tan x - \int \cos^3 x \sec x \tan x\, dx$

$= \log(\sec x) - \int \frac{1}{\sec^3 x} \sec x \tan x\, dx$

Put $u = \sec x$; $du = \sec x \tan x\, dx$

$$I = \log(\sec x) + \tfrac{1}{2}\cos^2 x + C$$

v) $I = \int \tan^4 x\, dx$

$= \int (\sec^2 x - 1)\tan^2 x\, dx = \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx$

$= \int \tan^2 x \sec^2 x\, dx - \tan x + x$

Take $\int \tan^2 x \sec^2 x\, dx$

Put $u = \tan x$; $du = \sec^2 x\, dx$

$$I = \tfrac{1}{3}\tan^3 x - \tan x + x + C$$

vi) $I = \int \sin 4x \cos 5x\, dx$

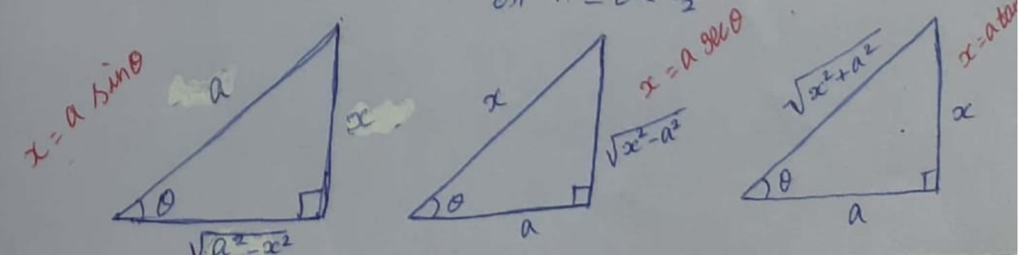
$= \frac{1}{2}\int(-\sin x + \sin 9x)\, dx = \frac{1}{2}\left[\cos x - \frac{1}{9}\cos 9x\right] + C$

vii) $I = \int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$

$= \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx = 1/\sqrt{2}$

TRIGONOMETRIC SUBSTITUTION:

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\ 0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$



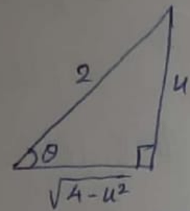
Evaluate i) $\int \frac{x}{\sqrt{3-2x-x^2}}dx$ ii) $\int \frac{1}{x^2\sqrt{x^2-1}}dx$ iii) $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}dx$

Soln.

i) Let $I = \int \frac{x}{\sqrt{3-2x-x^2}}dx$

$3-2x-x^2 = -(x^2+2x)+3$

$= -[(x+1)^2-1]+3 = -(x+1)^2+4$



$\cos\theta = \frac{\sqrt{4-u^2}}{2}$

Put $u = x+1 \Rightarrow x = u-1$

$du = dx$

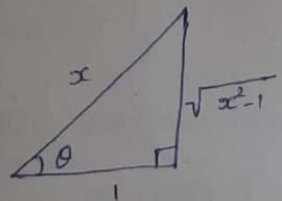
Put $u = 2\sin\theta \Rightarrow du = 2\cos\theta\, d\theta$; $\theta = \sin^{-1}(u/2)$

$\Rightarrow I = \int \frac{2\sin\theta - 1}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta\, d\theta = \int \frac{2\sin\theta-1}{2\cos\theta} \cdot 2\cos\theta\, d\theta$

$= -2\frac{\sqrt{4-u^2}}{2} - \sin^{-1}\left(\frac{u}{2}\right) + C$

$= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$ //

ii) Let $I = \int \frac{1}{x^2\sqrt{x^2-1}}dx$



Put $x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$\sin\theta = \frac{\sqrt{x^2-1}}{x}$

$\tan\theta = \sqrt{x^2-1}$

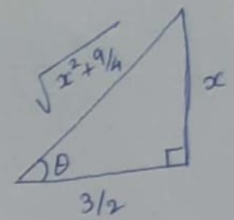
$I = \int \frac{1}{\sec^2\theta\tan\theta}\sec\theta\tan\theta\, d\theta$

$= \sin\theta + C$

$= \frac{\sqrt{x^2-1}}{x} + C$,

iii) Let $I = \int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}dx$

$= \int_0^{3\sqrt{3}/2} \frac{x^3}{8(x^2+9/4)^{3/2}}dx$



Put $x = \frac{3}{2}\tan\theta$ $\quad x \to 0 \Rightarrow \theta \to 0$

$dx = \frac{3}{2}\sec^2\theta\, d\theta$ $\quad x \to \frac{3\sqrt{3}}{2} \Rightarrow \theta \to \pi/3$

$I = \int_0^{\pi/3} \frac{1}{8} \frac{\frac{27}{8}\tan^3\theta}{\left(\frac{9}{4}\tan^2\theta + \frac{9}{4}\right)^{3/2}} \cdot \frac{3}{2}\sec^2\theta\, d\theta$

$= \int_0^{\pi/3} \left(\frac{1}{8}\right)\left(\frac{27}{8}\right)\left(\frac{3}{2}\right) \frac{\tan^3\theta\sec^2\theta}{\left(\frac{9}{4}\right)^{3/2}(1+\tan^2\theta)^{3/2}}d\theta$

$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^2\theta}{\sec\theta}\tan\theta\, d\theta = \frac{3}{16}\int_0^{\pi/3}[\sec\theta\tan\theta - \sin\theta]\,d\theta$

$= 3/32$ //

Integrate the following: i) $\int \frac{10}{(x-1)(x^2+9)}dx$ ii) $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$.

Soln:-

i) Let $I = \int \frac{10}{(x-1)(x^2+9)}dx$

$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \longrightarrow (1)$

$\Rightarrow A = 1, B = -1, C = -1$

$I = \int \frac{1}{x-1}dx - \int \frac{x}{x^2+9}dx - \int \frac{1}{x^2+9}dx$

$= \log(x-1) - \frac{1}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$ //

ii) $$I = \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$$

$$\begin{array}{r|l} & x+1 \\ \hline x^3 - x^2 - x + 1 & x^4 - 0x^3 - 2x^2 + 4x + 1 \\ & x^4 - x^3 - x^2 + x \\ & (-)\ (+)\ (+)\ (-) \\ \hline & x^3 - x^2 + 3x + 1 \\ & x^3 - x^2 - x + 1 \\ & (-)\ (+)\ (+)\ (-) \\ \hline & 4x \end{array}$$

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

$$= x + 1 + \frac{4x}{(x-1)^2(x+1)}$$

$\Rightarrow A = 1,\ B = 2,\ C = -1$

$$I = \frac{x^2}{2} + x - \frac{2}{x-1} + \log\left(\frac{x-1}{x+1}\right) + k.$$

RULE:

$$\int \frac{dx}{a\cos x + b\sin x} \quad (\text{or}) \quad \int \frac{dx}{a + b\sin x} \quad (\text{or}) \quad \int \frac{1}{a + b\cos x}\,dx$$

$$(\text{or}) \quad \int \frac{1}{a\sin x + b\cos x + c}\,dx$$

Put $t = \tan \frac{x}{2}$

$dt = \frac{1}{2}\sec^2 \frac{x}{2}\,dx$ $\qquad \sin x = \frac{2t}{1+t^2}$

$= \frac{1}{2}\left[1 + \tan^2 \frac{x}{2}\right]dx$ $\qquad \cos x = \frac{1-t^2}{1+t^2}$

$= \frac{1}{2}(1+t^2)\,dx$

1) Prove that $\displaystyle\int_0^{\pi} \frac{d\theta}{5 + 3\cos\theta} = \frac{\pi}{4}$

Soln.

Let $I = \displaystyle\int \frac{d\theta}{5 + 3\cos\theta}$

Put $t = \tan \frac{\theta}{2}$

$$I = \int \frac{\frac{2\,dt}{1+t^2}}{5 + 3\left(\frac{1-t^2}{1+t^2}\right)} = \int \frac{2\,dt}{8 + 2t^2}$$

$$= \int \frac{dt}{4 + t^2} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{\tan \theta/2}{2}\right)$$

$$\int_0^{\pi} \frac{d\theta}{5 + 3\cos\theta} = \left[\frac{1}{2}\tan^{-1}\left(\frac{\tan \theta/2}{2}\right)\right]_0^{\pi}$$

$$= \frac{1}{2}\tan^{-1}\infty = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

2) Prove that $\displaystyle\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{ab}$.

Soln.

Let $I = \displaystyle\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$$= \int \frac{dx}{\cos^2 x\,[a^2 + b^2\tan^2 x]} = \int \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$$

$$= \int \frac{dt}{b^2\left[t^2 + \left(\frac{a}{b}\right)^2\right]}$$

Put $t = \tan x$

$dt = \sec^2 x\,dx$

$$= \frac{1}{b^2}\left(\frac{b}{a}\right)\tan^{-1}\left(\frac{t}{a/b}\right) = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right)$$

$$\Rightarrow \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{ab}.$$

1) Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

Soln:

Gn. $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \log t = \infty$ (not finite)

$\therefore \int_1^\infty \frac{1}{x}\,dx$ is divergent.

2) Determine whether the integral $\int_1^\infty \frac{\log x}{x^2}\,dx$ is convergent or divergent.

Gn. $\int_1^\infty \frac{\log x}{x^2}\,dx$

Take $\int \frac{\log x}{x^2}\,dx$ Put $u = \log x \Rightarrow du = \frac{1}{x}\,dx$

$dv = \frac{1}{x^2}\,dx \Rightarrow v = -\frac{1}{x}$

$\int \frac{\log x}{x^2}\,dx = (\log x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\,dx\right)$

$= -\frac{1}{x}\left[\log x + 1\right]$

$\therefore \int_1^\infty \frac{\log x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\log x}{x^2}\,dx$

$= \lim_{t\to\infty}\left[-\frac{1}{x}(\log x + 1)\right]_1^t$

$= \lim_{t\to\infty}\left(-\frac{\log t}{t}\right) + 1 \longrightarrow (1)$

Take, $\lim_{t\to\infty}\left(-\frac{\log t}{t}\right)$

$\lim_{t\to\infty}\left(-\frac{\log t}{t}\right) = \frac{\infty}{\infty}$

By L' Hospital's rule

$= \lim_{t\to\infty} \frac{-(1/t)}{1} = 0$

$(1) \Rightarrow \int_1^\infty \frac{\log x}{x^2}\,dx = 1$ (finite) $\Rightarrow \int_1^\infty \frac{\log x}{x^2}\,dx$ is convergent.

3) Evaluate $\int_{-\infty}^\infty x e^{-x^2}\,dx$.

Soln:

Take $\int x e^{-x^2}\,dx$

Put $u = x^2 \Rightarrow du = 2x\,dx$.

$\int x e^{-x^2}\,dx = \int e^{-u}\,\frac{du}{2} = -\frac{1}{2}e^{-u} = -\frac{1}{2}e^{-x^2} \longrightarrow (1)$

$\int_{-\infty}^\infty x e^{-x^2}\,dx = \int_{-\infty}^0 x e^{-x^2}\,dx + \int_0^\infty x e^{-x^2}\,dx \longrightarrow (2)$

Take $\int_{-\infty}^0 x e^{-x^2}\,dx = \lim_{t\to-\infty}\int_t^0 x e^{-x^2}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_t^0$ [by (1)]

$= -\frac{1}{2}$

Take $\int_0^\infty x e^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t x e^{-x^2}\,dx = \frac{1}{2}$

$(2) \Rightarrow \int_{-\infty}^\infty x e^{-x^2}\,dx = 0$.

4) Evaluate $\int_0^3 \frac{1}{x-1}\,dx$

Soln.

Here, infinite discontinuity occurs at $x=1$.

$\therefore \int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$

$= \lim_{t\to1^-}\int_0^t \frac{dx}{x-1} + \lim_{t\to1^+}\int \frac{dx}{x-1}$

$= \lim_{t\to1^-}\log(t-1) + \lim_{t\to1^+}\log(t-1)$

$= -\infty + \infty = \infty$

$\therefore \int_0^3 \frac{1}{x-1}\,dx$ is divergent.

UNIT - V

MULTIPLE INTEGRALS

DOUBLE INTEGRATION IN CARTESIAN CO-ORDINATES:

1. Evaluate $\int_2^a \int_2^b \frac{dx\,dy}{xy}$.

Soln.

$$\int_2^a \int_2^b \frac{dx\,dy}{xy} = \log(b/2)\log(a/2).$$

2. $\int_0^5 \int_0^2 (x^2+y^2)\,dx\,dy$; Ans = $290/3$.

H.W 3) $\int_0^1 \int_0^{x^2} (x^2+y^2)\,dy\,dx$

DOUBLE INTEGRATION IN POLAR CO-ORDINATES

1) Evaluate $\int_0^{\pi/2} \int_0^{\sin\theta} r\,d\theta\,dr$.

Soln.

Let $I = \int_0^{\pi/2} \int_0^{\sin\theta} r\,dr\,d\theta$.

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

$$I_n = \int_0^{\pi/2} \cos^n\theta\,d\theta = \int_0^{\pi/2} \sin^n\theta\,d\theta$$

If n is odd, $I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}\cdot 1$

If n is even. $I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$

2) Evaluate $\int_0^{\pi}\int_0^{a} r\,dr\,d\theta$. Ans => $\frac{\pi a^2}{2}$.

H.W 3. Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta$; Ans => $\pi/4$

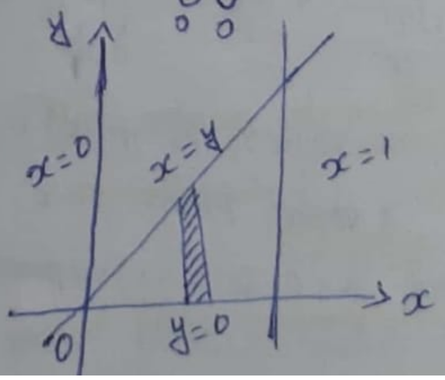
4) Sketch roughly the region of integration for $\int_0^1\int_0^x f(x,y)\,dy\,dx$

soln

Given: $\int_0^1\int_0^x f(x,y)\,dy\,dx$

x varies from 0 to 1

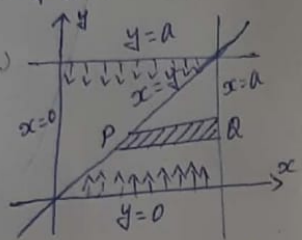
y varies from 0 to x



CHANGE OF ORDER OF INTEGRATION:

1) change of the order of $I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ and then evaluate.

Soln: The region of integration is bounded by $y=0$, $y=a$, $x=y$ and $x=a$.



Here, y varies from 0 to a, x varies from y to a.)

Here, $y=0$ to $y=a$ represents Horizontal path and $x=y$ to $x=a$ represents Horizontal strip sliding area.

Changing the order of integration is nothing but to change the Horizontal path into Vertical path and then to change the Horizontal Strip PQ into Vertical Strip RS.

Now, $x=0$ to a represents vertical path and $y=0$ to $y=x$ represents vertical strip RS sliding area.

Hence,
$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy = \int_0^a \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$$

$\because \int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}(x/a)$

$\tan^{-1}(1) = \pi/4$

$\tan^{-1}(0) = 0$

$$= \int_0^a \left[\tan^{-1}(y/x)\right]_0^x dx$$

$$= \frac{\pi}{4}\int_0^a dx = \frac{\pi}{4} a$$

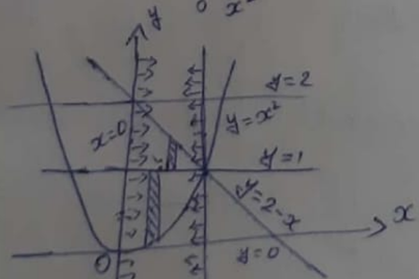
(a)

2) change the order of integration and hence evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$

Now, we divide the region into

$$I = I_1 + I_2$$

$$= \int_0^1 \int_{x^2}^{1} xy\, dy\, dx + \int_0^1 \int_1^{2-x} xy\, dy\, dx$$



In I_1, x varies from $x=0$ to $x=1$ and y varies from $y=x^2$ to $y=1$

Here, $x=0$ to $x=1$ represents vertical path and $y=x^2$ to $y=1$ represents vertical strip sliding area.

Changing the order of integration is nothing but to change the vertical path into Horizontal path and to change the vertical strip PQ into Horizontal Strip RS.

Hence, by changing the order, we get $\int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy$

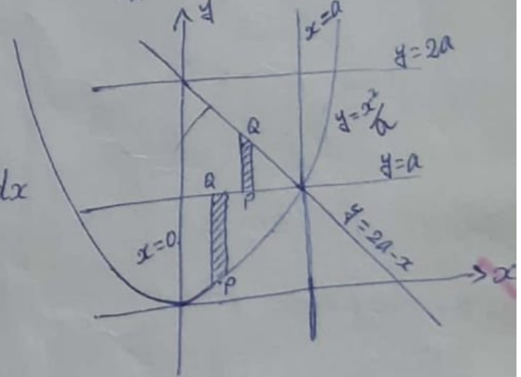
$$= \int_0^1 \left[y \frac{x^2}{2}\right]_0^{\sqrt{y}} dy = 1/6$$

$\|^{ly}$ In I_2

$$I_2 = \int_1^2 \int_0^{2-y} xy\, dx\, dy = 5/24 \quad ; \quad I = I_1 + I_2 = 3/8$$

(b)

3) Change the order of integration in $\int_0^a \int_{x^2/a}^{2a-x} xy\, dx\, dy$ and hence evaluate the same.



$$I = I_1 + I_2$$

$$I_1 = \int_0^a \int_{x^2/a}^{a} xy\, dy\, dx + \int_0^a \int_a^{2a-x} xy\, dy\, dx$$

$$I_1 = \int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy = \frac{a^4}{6}$$

$$I_2 = \int_a^{2a} \int_0^{2a-y} xy\, dx\, dy = \frac{5}{24}a^4 \quad ; \quad I = \frac{3}{8}a^4$$

4) change the order of integration in $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\, dx\, dy$ and then evaluate it.

Soln: The region of integration is bounded by $y=0$, $y=a$, $x=a-y$, $x=\sqrt{a^2-y^2}$.

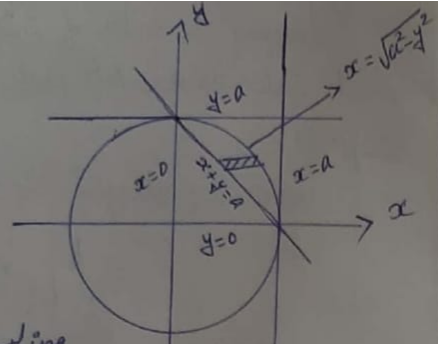
Here y varies from 0 to a

x varies from $a-y$ to $\sqrt{a^2-y^2}$.

H.W: change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\, dy$ and hence evaluate it.

(ie.,) $x+y=a$ to $x^2=a^2-y^2$

$\Rightarrow x+y=a$ to $x^2+y^2=a^2$



Here, $y=0$ to $y=a$ represents Horizontal path and $x=a-y$ to $x=\sqrt{a^2-y^2}$ represents Horizontal strip sliding area.

Changing the order of integration is nothing but to change the Horizontal path into vertical path and to change the horizontal strip into vertical strip.

Hence, by changing the order, we get

$$I=\int_0^a\int_{a-x}^{\sqrt{a^2-x^2}} y\,dy\,dx = \frac{a^3}{6}$$

CHANGE OF VARIABLES FROM CARTESIAN TO POLAR CO-ORDINATES:

1) By changing to polar co-ordinates, find the value of the integral

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dy\,dx.$$

Soln. Gn. $I=\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dy\,dx$

Cartesian form:

x varies from 0 to $2a$

y varies from 0 to $\sqrt{2ax-x^2}$

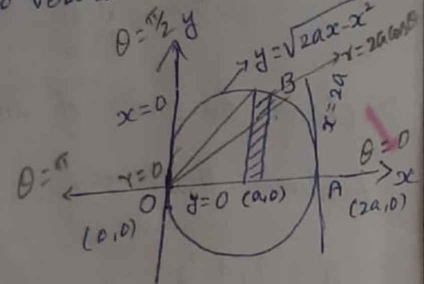
$\Rightarrow y^2=2ax-x^2$

$x^2+y^2=2ax.$

$\Rightarrow x^2+y^2-2ax=0 \longrightarrow (1)$

[Circle, centre $(a,0)$, radius $=a$]

Hence the region of integration is OAB



Polar form:

Let us transform this integral in Polar Co-ordinates by taking $x=r\cos\theta$, $y=r\sin\theta$, $dy\,dx=r\,dr\,d\theta$

For the region OAB,

θ varies from $\theta=0$ to $\theta=\pi/2$

r varies from $r=0$ to a point B on the circle

$(1)\Rightarrow x^2+y^2-2ax=0$

$r(r-2a\cos\theta)=0$

Hence r varies from 0 to $2a\cos\theta$

$$I=\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\cdot r\,dr\,d\theta$$

$$=4a^4\left(\tfrac{3}{4}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{2}\right)=\frac{3\pi}{4}a^4$$

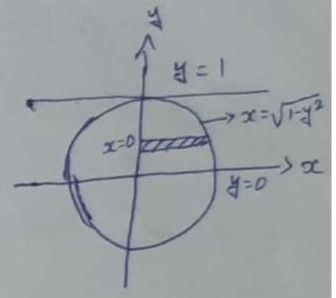
AREA ENCLOSED BY PLANE CURVES:

1) Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=1$.

Soln.

Gn. $x^2+y^2=1$

$x^2=1-y^2 \Rightarrow x=\pm\sqrt{1-y^2}$



Put $x=0$, $y^2=1 \Rightarrow y=\pm 1$

$\therefore$ x varies from $x=0$ to $\sqrt{1-y^2}$

y varies from $y=0$ to $y=1$

$\therefore$ The required area $=\int_0^1\int_0^{\sqrt{1-y^2}} xy\,dx\,dy = \frac{1}{8}$ square unit

2) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Soln.

$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}$

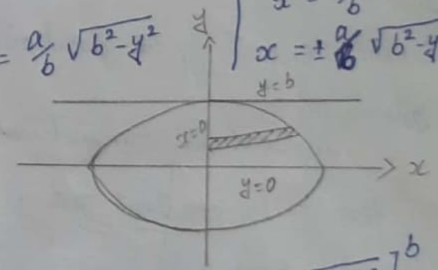
$x^2=\frac{a^2}{b^2}(b^2-y^2)$

$x=\pm\frac{a}{b}\sqrt{b^2-y^2}$

Area of ellipse $=4\times$ area of quadrant

x varies from $x=0$ to $x=\frac{a}{b}\sqrt{b^2-y^2}$

y varies from $y=0$ to $y=b$



$\therefore$ The required area $=4\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dx\,dy$

$$=\frac{4a}{b}\int_0^b\sqrt{b^2-y^2}\,dy=\frac{4a}{b}\left[\frac{b^2}{2}\sin^{-1}\frac{y}{b}+\frac{y}{2}\sqrt{b^2-y^2}\right]_0^b$$

$=\pi ab$ square units

3) Show that the area between the parabolas $y^2=4ax$ & $x^2=4ay$ is $\frac{16}{3}a^2$.

Soln. Gn. $y^2=4ax \longrightarrow (1)$

x	0	a	4a
$y=\pm2\sqrt{ax}$	0	$\pm 2a$	$\pm 4a$

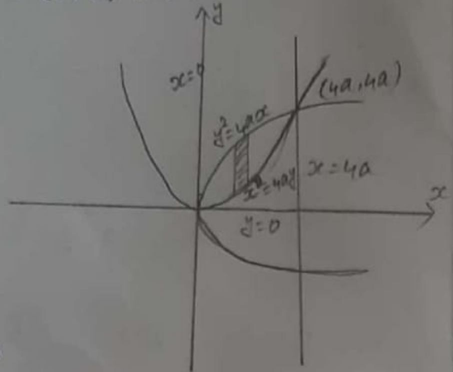
Gn. $x^2=4ay \longrightarrow (2)$

x	0	a	4a
$y=\frac{x^2}{4a}$	0	$a/4$	4a

$\therefore$ The point of intersection of (1) & (2) is $(0,0)$ & $(4a, 4a)$

x varies from $x=0$ to $x=4a$

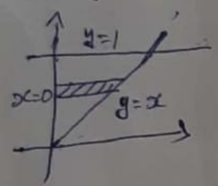
y varies from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$



$\therefore$ The required area $= \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$

$= \frac{16}{3} a^2$ square units

4) Find the area bounded by the lines $x=0$, $y=1$ & $y=x$ using double integration.



Soln. Gn:- $x=0$, $y=1$, $y=x$

Let $I = \iint_R dx\, dy = \int_0^1 \int_0^y dx\, dy = \frac{1}{2}$ square unit.

5) Find the area of the cardioid $r = a(1+\cos\theta)$, using a double integral.

Soln. The curve is symmetrical about the initial line.

Here, θ is constant and r varies from 0 to $a(1+\cos\theta)$. Then, θ varies from 0 to π.

Hence, required area $= 2\int_0^{\pi} \int_0^{a(1+\cos\theta)} r\, dr\, d\theta$

$= a^2 \int_0^{\pi} [1 + \cos^2\theta + 2\cos\theta]\, d\theta$

$= a^2 \int_0^{\pi} \left[1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\right] d\theta = \frac{3}{2} a^2 \pi$ sq. unit

H.W

1) Find the area inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

TRIPLE INTEGRALS:

1) Evaluate $I = \int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$.

Soln. $I = \int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$ [using Bernoulli formula

$= \int_0^{\log 2} \int_0^{x} [e^{2x+y} \cdot y - e^{x+y}]\, dy\, dx = \frac{8}{3}\log 2 - \frac{19}{9}$

2) Evaluate $\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2)\, dx\, dy\, dz$

Ans $= \frac{abc}{3}[a^2+b^2+c^2]$

3) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$. Ans $= \frac{\pi^2}{8}$.

$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a}$

(a)

VOLUME OF SOLIDS:

1. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.

Soln. Let $V = \iiint_V dz\, dy\, dx$

Here, z varies from $z=0$ to $z=4-y$

y varies from $y = -\sqrt{4-x^2}$ to $y = \sqrt{4-x^2}$

x varies from $x=-2$ to $x=2$.

$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-y} dz\, dy\, dx = 16\int_0^2 \sqrt{4-x^2}\, dx = 16\pi$ cubic units

$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2}\right]$

2) Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

Soln.

$V = 8 \times$ Volume in an octant.

z varies from $z=0$ to $z=\sqrt{a^2-x^2-y^2}$

y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

x varies from $x=0$ to $x=a$.

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} + \frac{y}{2}\sqrt{a^2-x^2-y^2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$= \frac{4}{3}\pi a^3$ cubic units.

3) Find the volume of that portion of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in the first octant using triple integration.

Soln. Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \longrightarrow (1)$ [First octant] Ans $= \frac{\pi abc}{6}$ cubic units

Volume $= \iiint dz\,dy\,dx$

NOTE: volume of the whole ellipsoid $= 8\left(\frac{\pi abc}{6}\right) = \frac{4}{3}\pi abc$.

z varies from $z=0$ to $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

y varies from $y=0$ to $y=b\sqrt{1-x^2/a^2}$

x varies from $x=0$ to $x=a$.

$$\text{Volume} = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$= c\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{b^2(1-\frac{x^2}{a^2})-y^2}{b^2}}\,dy\,dx = \frac{c}{b}\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\left(\sqrt{b^2(1-\frac{x^2}{a^2})}\right)^2-y^2}\,dy\,dx$$

H.W

4) Find the volume of the region bounded by the paraboloid $z=x^2+y^2$ and the plane $z=4$.

5) Evaluate $\iiint\limits_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region of integration bounded by the planes $x=0, y=0, z=0, x+y+z=1$. Ans: $\frac{1}{2}\log 2 - \frac{5}{16}$

6) Evaluate the integration $\iiint xyz\,dx\,dy\,dz$ taken throughout the volume for which $x,y,z \ge 0$ and $x^2+y^2+z^2 \le 9$.

Soln.

x varies from 0 to 3

y varies from 0 to $\sqrt{9-x^2}$

z varies from 0 to $\sqrt{9-y^2-x^2}$

$$\iiint xyz\,dx\,dy\,dz = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-y^2-x^2}} xyz\,dz\,dy\,dx = \frac{243}{16}$$

H.W same problem: Using spherical polar co-ordinates

Let $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

1) Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

2) Find the area inside the circle $r=a\sin\theta$ and outside the cardioid $r=a(1-\cos\theta)$

ORDINARY DIFFERENTIAL EQUATIONS.

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS:

(a) General form of a linear differential equation of the n^{th} order with constant coefficients is

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = X \longrightarrow (1)$$

where $k_1, k_2, \dots k_n$ are constants.

(b)(i) The general form of the linear differential equation of second order is

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R$$

where P and Q are constants and R is a function of x or constant.

(ii) Differential Operators:

The symbol D stands for the operation of differential

(ie.,) $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$

$\Rightarrow$ $1/D$ stands for the operation of integration

$\Rightarrow$ $1/D^2$ stands for the operation of integration twice.

(iii) Complete soln is

$$y = C.F + P.I$$

(iv) To find the complementary functions

Roots of A.E	C.F
(i) Roots are real & different m_1, m_2	$Ae^{m_1 x} + Be^{m_2 x}$
(ii) Roots are real & equal $m_1 = m_2 = m$ (say)	$(Ax+B)e^{mx}$
(iii) Roots are imaginary $\alpha \pm i\beta$	$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

R.H.S = 0

1) Solve : $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0.$

$$m = 3 \pm 2i$$

2) Solve $(D^2+1)y = 0$, given $y(0)=0$, $y'(0)=1$

$$A = 0, \quad B = 1$$

3) Solve: $(D^3+D^2+4D+4)y = 0.$

$$m = -1, \pm 2i$$

4) Solve $(D^3-3D^2+4D-2)y = e^x$

$$m = 1, 1\pm i$$

4) Solve $(D^2+7D+12)y = 14e^{-3x}$

R.H.S = e^{ax} [Replace D by a]

1) Solve $(D^2+6D+5)y = e^{2x}$

$m = -1, -5$; $P.I = \frac{1}{21}e^{2x}$

2) Find the particular integral of $(D^2-4D+4)y = 2^x$

$$(D-2)^2 y = 2^x = e^{\log 2^x} = e^{x\log 2} = e^{(\log 2)x}$$

$$P.I = \frac{1}{(\log 2 - 2)^2} e^{(\log 2)x}$$

3) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$

Soln.

$$m = -2 \pm i$$

$$P.I = -\frac{e^x}{10} - \frac{e^{-x}}{2}$$

$$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$$

R.H.S = $\cos ax$ (or) $\sin ax$ [Replace D^2 by $-a^2$]

1) Find the P.I of $(D^2+4)y = \cos 2x$

$$P.I = \frac{x}{4}\sin 2x$$

2) Solve $\frac{d^2y}{dx^2} + 4y = \sin 2x.$

$$y = A\cos 2x + B\sin 2x - \frac{x}{4}\cos 2x$$

3) Find the P.I of $(D^2+1)y = \sin x \sin 2x$

$$P.I = \frac{1}{16}\cos 3x + \frac{x}{4}\sin x$$

4) Find the P.I of $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x.$

$$\Rightarrow P.I = -\frac{x}{8}\sin 2x$$

R.H.S = x^n

Formulae:

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

5) Solve $(D^2+2D+2)y = e^{-2x} + \cos 2x$

$$\Rightarrow y = e^{-x}(A\cos x + B\sin x) + \frac{1}{2}e^{-2x} + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$$

6) Solve $(D^3-2D^2+4D-8)y = e^{2x} + \sin x\cos x$

$$= e^{2x} + \frac{1}{2}\sin 2x$$

$$\Rightarrow m = 2, \pm 2i$$

$$P.I = \frac{x}{8}e^{2x} + \frac{x}{32}(\cos 2x - \sin 2x)$$

1) Solve $(D^2-1)y = x.$

Soln.

$m = \pm 1 \quad \Rightarrow C.F = Ae^{-x} + Be^{x}$

$$P.I = -x$$

2) $(D^2-3D+2)y = 2\cos(2x+3) + 2e^x$

$$y = C_1e^x + C_2e^{2x} - \frac{1}{10}[3\sin(2x+3) + \cos(2x+3)] - 2xe^x$$

3) Solve $(D^4-2D^3+D^2)y = x^3$

$\Rightarrow m = 0, 0, 1, 1. \quad \Rightarrow P.I = \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$

R.H.S $= e^{ax} X$ [Replace D by $D+a$]

1) Find the P.I of $(D^2+1)y = xe^x$.

$$P.I = \frac{1}{D^2+1} x e^x = e^x \frac{1}{(D+1)^2+1} x$$

$$= e^x \frac{1}{D^2+2D+2} x = \frac{e^x}{2}\left[1+\frac{D^2+2D}{2}\right]^{-1} x$$

$$= \frac{e^x}{2}(x-1)$$

H.W

1) Find the P.I of $(D^2+2D+1)y = e^{-x}x^2$.

2) Find the P.I of $(D^2-2D+1)y = e^x(3x^2-2)$

3) Solve $(D^3-7D-6)y = (1+x)e^{2x}$.

$\Rightarrow m = -1, -2, 3$

$P.I = -\frac{e^{2x}}{12}\left(x+\frac{17}{12}\right)$

2) Solve $(D^2+5D+4)y = e^{-x}\sin 2x$

Soln.

$m = -1, -4$

$$P.I = \frac{1}{D^2+5D+4} e^{-x}\sin 2x$$

$$= e^{-x}\frac{1}{3D-4}\sin 2x = e^{-x}\frac{3D+4}{9D^2-16}\sin 2x$$

$$= \frac{-e^{-x}}{26}[3\cos 2x + 2\sin 2x]$$

3) Find the P.I of $(D-1)^2 y = e^x \sin x$

$P.I = -e^x \sin x$

R.H.S $= x^n \sin ax$ (or) $x^n \cos ax$

4) Solve $(D^2+4)y = x^2\cos 2x$.

C.F $= A\cos 2x + B\sin 2x$

$$P.I = \frac{1}{D^2+4} \text{ R.P of } x^2 e^{i2x}$$

$$= \text{R.P of } e^{i2x}\frac{1}{D^2+4iD}x^2 = \frac{x^2}{16}\cos 2x + \frac{x^3}{12}\sin 2x - \frac{x}{32}\sin 2x$$

$$\frac{1}{D^2+4iD} = \frac{1}{4iD}\frac{1}{\left(1+\frac{D^2}{4iD}\right)}$$

$$= \frac{1}{4iD}\left[1-\frac{iD}{4}\right]^{-1}$$

$$= \frac{-i}{4D}\left(1-\frac{iD}{4}\right)^{-1}$$

H.W

1) Solve $(D^2-4D+4)y = e^{2x} + \cos 4x + x^2$.

2) Solve $(D^2+5D+4)y = e^{-x}\sin 2x$.

3) Solve $(D^2-3D+2)y = xe^{3x} + \sin 2x$.

METHOD OF VARIATION OF PARAMETERS:

The second order eqn $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = X \longrightarrow (1)$

[where 'X' is a function of x]

The complementary function of (1)

$$C.F = c_1 f_1 + c_2 f_2 \longrightarrow (2)$$

where c_1, c_2 are constants and f_1 and f_2 are functions of x.

Then $P.I = Pf_1 + Qf_2 \longrightarrow (3)$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}dx \longrightarrow (4)$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}dx \longrightarrow (5)$$

$$\therefore y = c_1 f_1 + c_2 f_2 + P.I$$

NOTE:

The Wronskian of f_1, f_2 of (1) is given by

$$W = \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix} = f_1 f_2' - f_2 f_1'$$

1) Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ by using method of variation of parameters.

Soln.

$C.F = C_1 \cos x + C_2 \sin x$

$= C_1 f_1 + C_2 f_2 \quad ; \quad f_1 f_2' - f_2 f_1' = 1$

$P = -x \quad ; \quad Q = \int \frac{\cos x}{\sin x}\, dx = \log(\sin x)$

$P.I = -x \cos x + \log(\sin x) \sin x$

2) Solve $(D^2 + a^2)y = \tan ax$ by the method of variation of parameters.

Soln.

$C.F = C_1 \cos ax + C_2 \sin ax$

$f_1 f_2' - f_1' f_2 = a$

$P = -\frac{1}{a} \int \frac{\sin^2 ax}{\cos ax}\, dx = -\frac{1}{a} \int \frac{1-\cos^2 ax}{\cos ax}\, dx$

$= -\frac{1}{a} \left[\frac{1}{a} \log(\sec ax + \tan ax) - \frac{\sin ax}{a} \right]$

$= \frac{1}{a^2} \left[\sin ax - \log(\sec ax + \tan ax) \right]$

$Q = -\frac{1}{a^2} \cos ax$

3) Solve $(D^2+4)y = \sec 2x$ by the method of variation of parameters

soln.

$C.F = C_1 \cos 2x + C_2 \sin 2x$

$f_1 f_2' - f_1' f_2 = 2$

$P = -\frac{1}{2} \int \tan 2x\, dx = -\frac{1}{2} \left[-\frac{\log(\cos 2x)}{2} \right] = \frac{1}{4} \log(\cos 2x)$

$Q = \frac{1}{2} x$

H.W

4) Solve: $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

5) Solve $(D^2 - 4D + 4)y = e^{2x}$

6) Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x \cot x$

using the method of variation of parameters.

CAUCHY'S AND LEGENDRE'S LINEAR EQUATIONS:

HOMOGENEOUS EQUATIONS OF EULER TYPE [CAUCHY'S TYPE]

<u>Linear Differential Equations with Variable Coefficients</u>

An eqn of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = f(x) \quad \to (1)$$

where $a_1, a_2, \dots a_n$ are constants and $f(x)$ is a function of x.

Let $x = e^z$ (or) $z = \log x$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$

$x \frac{dy}{dx} = D'y \quad$ where $D' = \frac{d}{dz}$

IIIly $\quad x^2 \frac{d^2y}{dx^2} = D'(D'-1)y \quad ; \quad x^3 \frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y \dots$

1) Solve $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0$

Soln.

Gn. $(x^2 D^2 - xD + 1)y = 0$

Put $x = e^z$, $\log x = z \quad ; \quad xD = D'$, $x^2 D^2 = D'(D'-1)$

$\Rightarrow [D'^2 - 2D' + 1]y = 0 \qquad y = (Az + B)e^z$

$\qquad = x(A \log x + B)$

$m = 1, 1$

2) Solve $x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

Multiply by x

$\Rightarrow (x^2 D^2 + x D) y = 0$

$D'^2 y = 0$

$m = 0, 0$

$y = Az + B$

$= A(\log x) + B$.

3) Solve : $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^2 + \cos(\log x)$

Soln.

$(D'^2 - 4D' + 4) y = e^{2z} + \cos z$

$m = 2, 2$

$C.F = (A \log x + B) x^2$

$P.I_1 = \frac{1}{(D'-2)^2} e^{2z}$

$= \frac{x^2}{2} (\log x)^2$

$P.I_2 = \frac{1}{D'^2 - 4D' + 4} \cos z$

$= \frac{1}{25} [3\cos(\log x) - 4\sin(\log x)]$

4) Solve : $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = \frac{12 \log x}{x^2}$

Soln.

$(x^2 D^2 + x D) y = 12 \log x$

$(D'^2) y = 0$

$m = 0, 0$

$P.I = \frac{1}{(D')^2} 12z$

$= 2z^3$

$= 2(\log x)^3$

4) Reduce $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 3y = x$ into a differential eqn with Constant coefficients.

Soln.

$(D'^2 - 4D' + 3) y = e^z$.

H.W

5) Solve $[x^2 D^2 - x D + 1] y = \left[\frac{\log x}{x}\right]^2$.

6) Transform $(x^2 D^2 + x D + 1) y = 0$ into differential eqn with Constant coefficients, where $D = \frac{d}{dx}$.

LEGENDRE'S LINEAR DIFFERENTIAL EQUATION:

An eqn of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = Q$$

where k's are constants and Q is a function of x is called Legendre's linear differential eqns.

Put $ax + b = e^z$

ie., $z = \log(ax+b)$

If $D' = \frac{d}{dz}$, then $(ax+b) D = a D'$

$(ax+b)^2 D^2 = a^2 D'(D'-1)$ and so on.

1) Solve : $[(x+1)^2 D^2 + (x+1) D + 1] y = 4 \cos[\log(x+1)]$.

Soln.

Put $x + 1 = e^z$

$z = \log(x+1)$

$(x+1) D = D'$

$(x+1)^2 D^2 = D'(D'-1)$

$[D'^2 + 1] y = 4 \cos z$

$m = \pm i$

$C.F = A \cos[\log(x+1)] + B \sin[\log(x+1)]$

$P.I = 2 \log(x+1) \sin(\log(x+1))$,,

2) Solve : $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Soln.

Put $3x+2 = e^z$ Let $(3x+2)D = 3D'$

$\log(3x+2) = z$ $(3x+2)^2 D^2 = 9D'(D'-1)$

$x = \frac{1}{3}e^z - \frac{2}{3}$

$\Rightarrow (9D'^2 - 36)y = \frac{1}{3}e^{2z} - \frac{1}{3}$

$\div 9$

$\Rightarrow (D'^2 - 4)y = \frac{1}{27}e^{2z} - \frac{1}{27}$

$m = \pm 2$

$P.I_1 = \frac{\log(3x+2)}{108}(3x+2)^2$

$P.I_2 = +\frac{1}{108}$

3) Transform the eqn $(2x+3)^2\frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$ into a differential eqn with constant coefficients.

Soln.

$(4D'^2 - 8D' - 12)y = 3e^z - 9$

$\div 4$

$(D'^2 - 2D' - 3)y = \frac{1}{4}(3e^z - 9)$.

H.W

4) Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

5) Solve $(x+2)^2\frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x + 4$

SIMULTANEOUS FIRST ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

1) Solve : $\frac{dx}{dt} + y = e^t$, $x - \frac{dy}{dt} = t$

Soln.

$(1) - (2)\times D \Rightarrow (D^2+1)y = e^t - 1 = e^t - e^{0t}$

$m = \pm i$

$C.F = A\cos t + B\sin t$

$P.I_1 = \frac{1}{2}e^t$; $P.I_2 = 1$

$y = A\cos t + B\sin t + \frac{1}{2}e^t + 1$

$Dy = -A\sin t + B\cos t + \frac{1}{2}e^t$

$(2) \Rightarrow x = Dy + t$

$= -A\sin t + B\cos t + \frac{1}{2}e^t + t$ ..

2) Solve : $\frac{dx}{dt} + 2y = 5e^t$ and $\frac{dy}{dt} - 2x = 5e^t$ given that $x = -1$ and $y = 3$ at $t = 0$.

Soln.

$m = \pm 2i$

$y = A\cos 2t + B\sin 2t + 3e^t$

$x = -A\sin 2t + B\cos 2t - e^t$

Gn. $x(0) = -1$ & $y(0) = 3$

$A = 0$, $B = 0$

Hence, the eqns are

$x = -e^t$; $y = 3e^t$

3) Solve : $Dx + y = \sin 2t$; $-x + Dy = \cos 2t$

Soln.
$$(D^2+1)x = \cos 2t$$
$$m = \pm i$$

C.F $= A\cos t + B\sin t$

P.I $= -\frac{1}{3}\cos 2t$

$x = A\cos t + B\sin t - \frac{1}{3}\cos 2t$

$y = A\sin t - B\cos t + \frac{1}{3}\sin 2t$.

H.W

4) Solve: $Dx + y = \sin t$, $x + Dy = \cos t$ given that $x = 2$ and $y = 0$ at $t = 0$.

5) Solve : $\frac{dx}{dt} + 2x - 3y = t$; $\frac{dy}{dt} - 3x + 2y = e^{2t}$.

Soln.
$$(D^2+4D-5)y = 3t + 4e^{2t}$$
$$m = 1, -5$$
$$y = Ae^{t} + Be^{-5t} - \frac{3}{5}t - \frac{12}{25} + \frac{4}{7}e^{2t}$$
$$x = Ae^{t} - \frac{3}{2}Be^{-5t} - \frac{2}{5}t + \frac{3}{7}e^{2t} - \frac{13}{25}$$

METHOD OF UNDETERMINED COEFFICIENTS:

S.NO	FUNCTION X	CHOICE OF P.I
1.	ke^{px}	Ce^{px}
2.	$k\sin(ax+b)$ (or) $k\cos(ax+b)$	$C_1\sin(ax+b) + C_2\cos(ax+b)$
3.	$ke^{px}\sin(ax+b)$ (or) $ke^{px}\cos(ax+b)$	$C_1e^{px}\sin(ax+b) + C_2e^{px}\cos(ax+b)$
4	kx^m where $m = 0, 1, 2, \dots$	$C_0 + C_1x + C_2x^2 + \dots + C_mx^m$

1) Solve $(D^2 - 3D + 2)y = 6e^{3x}$

Soln. Gn. $y'' - 3y' + 2y = 6e^{3x}$ $\longrightarrow (1)$

The A.E is $m^2 - 3m + 2 = 0$

$\Rightarrow m = 1, 2$

C.F $\Rightarrow y_c = Ae^{x} + Be^{2x}$

Here, the soln set $S = \{e^{x}, e^{2x}\}$

R.H.S of (1) is not a member of S

Choose, P.I $\Rightarrow y_p = Ce^{3x}$ $\longrightarrow (2)$

$y_p' = 3Ce^{3x}$, $y_p'' = 9Ce^{3x}$

(1) $\Rightarrow$ $C = 3$

(2) $\Rightarrow$ $y_p = 3e^{3x}$

$\therefore y = y_c + y_p$

2) Solve : $y'' + 6y' + 5y = 2e^{x} + 10e^{5x}$ $S = \{e^{-x}, e^{-5x}\}$

Soln. $\Rightarrow m = -1, -5$

$y_c = Ae^{-x} + Be^{-5x}$

Choose $y_p = C_1e^{x} + C_2e^{5x}$

$\Rightarrow C_1 = \frac{1}{6}$, $C_2 = \frac{1}{6}$

3) Solve $y'' - y = e^{x} \sin 2x$

choose P.I $\Rightarrow y_p = C_1 e^{x} \sin 2x + C_2 e^{x} \cos 2x$

$C_1 = C_2 = -\frac{1}{8}$ $\quad y_p'' = C_1[4e^{x}\cos 2x - 3e^{x}\sin 2x] + C_2[-4e^{x}\sin 2x - 3e^{x}\cos 2x]$

4) Solve $y'' - 9y = x^3 + e^{2x} - \sin 3x$.

choose. $y_p = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 e^{2x} + C_5 \sin 3x + C_6 \cos 3x$

$\Rightarrow C_0 = 0,\ C_1 = -\frac{2}{27},\ C_2 = 0,\ C_3 = -\frac{1}{9}$

$C_4 = -\frac{1}{5},\ C_5 = \frac{1}{18},\ C_6 = 0$

5) Solve: $\frac{d^2y}{dx^2} + 9y = \cos 3x$.

Gn: $y'' + 9y = \cos 3x \longrightarrow (1)$

$\Rightarrow y_c = A \cos 3x + B \sin 3x$

Here. the soln. set $S = \{\cos 3x, \sin 3x\}$

choose. $y_p = C_1 \sin 3x + C_2 \cos 3x$.

R.H.S of (1) is a member of S.

The corresponding terms should be multiplied by x.

$y_p = x[C_1 \sin 3x + C_2 \cos 3x]$

$\Rightarrow y_p = \frac{x}{6} \sin 3x$

6) Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x} + \sin x$

Gn: $y'' - 5y' + 6y = e^{3x} + \sin x \longrightarrow (1)$

$y_c = Ae^{2x} + Be^{3x}$

Here. the soln set $S = \{e^{2x}, e^{3x}\}$

choose $y_p = C_1 e^{3x} + C_2 \sin x + C_3 \cos x$

First term in R.H.S of (1) is a member of S.

Here, the corresponding term should be multiplied by x

$y_p = C_1 x e^{3x} + C_2 \sin x + C_3 \cos x$

$\Rightarrow C_2 = \frac{1}{10} = C_3$.

$\therefore y_p = x e^{3x} + \frac{1}{10} \sin x + \frac{1}{10} \cos x$